① ② ③ ④ ⑤ ⑥

$a_{00} + a_{10}x_1 + a_{01}x_2 + a_{20}x_1^2 + a_{02}x_2^2 + a_{11}x_1x_2 + a_{30}x_1^3 + \dots + a_{03}x_2^3 + a_{40}x_1^4 + \dots + a_{04} + a_{50}x_1^5 + \dots + a_{05}$

Department of Applied Mechanics, Indian Institute of Technology Delhi

Second Semester, 2022-2023

APL 311 Introduction to Finite Element Method

Minor II (Open Book)

Date: 24/03/2023 Duration: 60 min Maximum Marks: 15

Problem

Show that a function $v \in P_5'(K) = \{v \in P_5(K) : \frac{\partial v}{\partial n}$ is a polynomial of degree at most 3 on each side of $K\}$ is uniquely determined by the following degrees of freedom

$$D^\alpha v(a^i); |\alpha| \leq 2; i = 1, 2, 3$$

where a^i are the vertices of the triangle K.

$D^1 v = \frac{\partial v}{\partial x_1}$, $\partial v / \partial x_2$

$\frac{\partial v}{\partial x_1} \cdot \frac{\partial x_1}{\partial n} + \frac{\partial v}{\partial x_2} \frac{\partial x_2}{\partial n}$

Solution

Since the dim $P_5'(K)$ is equal to the number of degrees of freedom,

Q. 1 How many degrees of freedom are there? Mention them explicitly. [2 marks, no partial marks]

it is sufficient to prove that if all the above degrees of freedom are zero, then $v \equiv 0$.

Q. 2 Why is it sufficient to prove the above for the overall objective? [1 mark]

We first note that if s denotes the direction from a^2 to a^3, then

$$v(a^i) = \frac{\partial v}{\partial s}(a^i) = \frac{\partial^2 v}{\partial s^2}(a^i) = 0; i = 2, 3$$

Q. 3 Mathematically justify the above first and the second derivatives being equal to zero. [1+2 marks]

Together with the fact that $v(a^2) = v(a^3)$, this shows that v vanishes along the side a^2a^3 since v is a polynomial of degree at most 5.

Q. 4 Provide a mathematical explanation (no diagrams/figures) for the above argument. [3 marks]

Further, given that $\frac{\partial v}{\partial n}$ is a polynomial of degree at most 3 on side a^2a^3, this implies

$$\frac{\partial v}{\partial n} = 0 \quad \text{on side } a^2a^3$$

Q. 5 Mathematically justify (no diagrams/figures) $\frac{\partial v}{\partial n} = 0$ on side a^2a^3. [4 marks]

Thus, both v and $\frac{\partial v}{\partial n}$ vanish on a^2a^3 which means we may "factor our" $(\lambda_1(x))^2$ out of $v(x)$ as

$$v(x) = (\lambda_1(x))^2 v_3(x) \quad \text{for} x \in K$$

where $v_3(x) \in P_3(K)$. In the same way, we see that we may "factor out" $(\lambda_2(x))^2$ and $(\lambda_3(x))^2$ for the other two sides. Therefore

$$v(x) = \gamma(\lambda_1(x))^2(\lambda_2(x))^2(\lambda_3(x))^2$$

where $\gamma \in R$. But $v \in P_5(K)$ and the only possibility is that $\gamma \equiv 0$ such that $v \equiv 0$ on K. Hence proved.

Q. 6 Further demonstrate that the above finite element space $V_h \subset C^1(\bar{\Omega})$. [2 marks]