APL 701 Continuum Mechanics
Department of Applied Mechanics
Indian Institute of Technology Delhi

Arjun Sharma

Major (Total: 30 points)
Each question has same weightage

(1) The stress distribution in a thin, light circular disk with density ρ rotating at angular speed ω is to be determined (for example, a rotating compact disk in a computer). The internal radius of the disk is R_1 and the external radius is R_2.

(a) Provide the equations of equilibrium in radial coordinates after setting the radial body force term as $\rho\omega^2 r$. Reduce the equations after making the following assumptions (i) The stress, strain and displacement components are independent of azimuthal coordinate, ϕ and (ii) the only non-zero displacement component is the radial displacement, $u_r(r)$.

(b) Substitute the stress-strain relation for a linearly elastic isotropic solid and the strain displacement relations to obtain an ordinary differential equation (of second degree) in terms of displacement component $u_r(r)$.

(c) Obtain a general solution for the above differential equation. Since the differential equation is of second degree, two boundary conditions are needed to specify the constants. The stress at the internal surface of the disk is specified as $\sigma_{rr} = \sigma_0$ and at the external surface as $\sigma_{rr} = 0$. Determine the exact solution for the given system with the given boundary conditions.

(2) A viscous clutch is to be made from a pair of closely spaced parallel disks enclosing a thin layer of viscous fluid (Newtonian fluid). Consider the flow to be steady, incompressible and purely circumferential.

(a) Develop algebraic expressions, for the torque and the power transmitted by the disk pair in terms of viscosity μ, disk radius R, disk spacing a, and the angular speeds ω_i of the input disk and ω_o of the output disk.

(b) Develop expressions for the slip ratio $s = \frac{\Delta\omega}{\omega_i}$ in terms of ω_i and the torque transmitted.

(c) Determine the efficiency η in terms of the slip ratio.

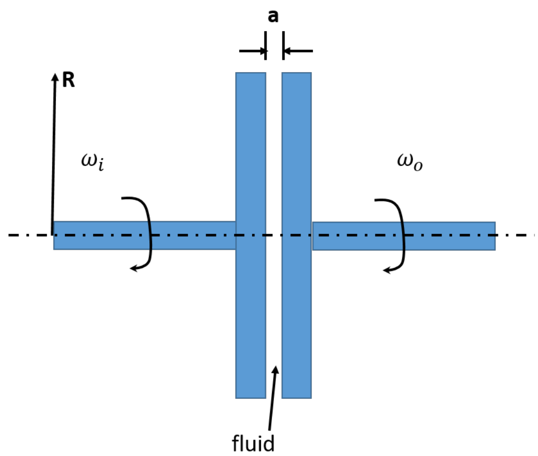

Figure 1: Question 2

(3) (a) The stress components in a thin plate bounded by $x_1 = \pm L$ and $x_2 = \pm h$ are given by,

$$\sigma_{11} = Wm^2 \cos\left(\frac{\pi x_1}{2L}\right)\sinh(mx_2)$$
$$\sigma_{22} = -\frac{W\pi^2}{4L^2}\cos\left(\frac{\pi x_1}{2L}\right)\sinh(mx_2)$$
$$\sigma_{12} = \frac{W\pi m}{2L}\sin\left(\frac{\pi x_1}{2L}\right)\cosh(mx_2)$$
$$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$$

where W and m are constants.
(i) Find the tractions and total force on the edges, $x_2 = h$ and $x_1 = -L$. Determine the net bending moment about the origin ($x_1 = 0, x_2 = 0$) due to traction on the edge, $x_2 = h$.
(ii) Find the principal stress components and principal axes of stress at $(0, h, 0)$ and at $(L, 0, 0)$.

(b) A cylinder whose axis is parallel to the x_3 axis and whose normal cross section is the square $-a \leq x_1 \leq a$, $-a \leq x_2 \leq a$, is subjected to torsion by couples acting over its ends $x_3 = 0$ and $x_3 = L$. The stress components are given by, $\sigma_{13} = \frac{\partial\psi}{\partial x_2}$, $\sigma_{23} = -\frac{\partial\psi}{\partial x_1}$, $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{12} = 0$, where $\psi = \psi(x_1, x_2)$ is the stress function.
(i) Show that these stress components satisfy the equilibrium equations with no body forces.
(ii) Show that the difference between the maximum and minimum principal stress components is $2\left[\left(\frac{\partial\psi}{\partial x_1}\right)^2 + \left(\frac{\partial\psi}{\partial x_2}\right)^2\right]^{1/2}$, and find the principal axis which corresponds to the zero principal stress component.
(iii) For the case, $\psi = (x_1^2 - a^2)(x_2^2 - a^2)$ show that the lateral surfaces are free from traction and that the couple acting on each end face is $\frac{32a^6}{9}$.

$\sinh x = \frac{\exp(x)-\exp(-x)}{2}$, $\cosh x = \frac{\exp(x)+\exp(-x)}{2}$
Constitutive relation for Newtonian incompressible fluid: $\sigma_{ij} = -p\delta_{ij} + 2\mu\varepsilon_{ij}$
Constitutive relation for linear isotropic elastic solid: $E_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right]$
Equilibrium equations: $\frac{\partial\sigma_{ij}}{\partial x_j} + \rho b_i = 0$ for $i = 1, 2, 3$ where b_i are the volumetric body forces
Momentum equations (in the azimuthal and axial directions) for incompressible fluid flow with constant density and viscosity:

$$\frac{Dv_\phi}{Dt} + \frac{v_\phi v_r}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\phi} + \nu\left(\nabla^2 v_\phi - \frac{v_\phi}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial\phi}\right)$$
$$\frac{Dv_z}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 v_z$$
$$\text{where } \nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$