CLL 231: Fluid Mechanics for Chemical Engineers

Major Examination

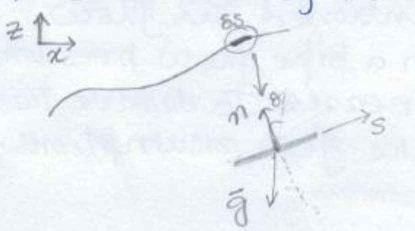
Date:11-April-2022 (Schedule: 11:15 am to 1:15 pm)

Total Marks: 35 **Duration: 2h 00 min**

Q1. Derive the Bernoulli's equation along the streamline direction for a steady, incompressible, and inviscid flow. (**Marks: 7**)

Answer: (Option1)

For the derivation of Bernoulli' equation, we will consider one streamline and continue with Newton's second Law of motion. To derive the equation, we will start with Newton's second law without any shear component (inviscid). Here we are going to use two sets of co-ordinates system. x-z co-ordinate for the entire flow field (2-D) and s-n coordinate at each point along the streamline. s-n coordinate changes if the direction of the streamline changes.



description of co-ordinate along a streamline

So the motion of each point along the streamline, $s = S(t)$
$s \to$ co-ordinate along the streamline and $n \to$ normal to the stream line. Now recall the governing equation which we obtained from Newton's second law (inviscid) that is

$$\boxed{-\bar{\nabla} P - \rho\bar{g} = \rho\bar{a}}$$ vector form

We will now re-write this equation along s-direction. We need to findout the components of each term along s-direction. Contribution from pressure term will be $\frac{\partial P}{\partial s}$. Now '$\bar{g}$' is acting along negative z-direction which is making an angle 'θ' with the local vertical co-ordinate 'n'. So, the contribution of '$\bar{g}$' along s direction is '$g\sin\theta$'. Finally, we need to find out the acceleration term along 's'. From the definition of acceleration we get,

$$\bar{a} = \frac{d\bar{V}}{dt} = a_s\hat{s} + a_n\hat{n}$$

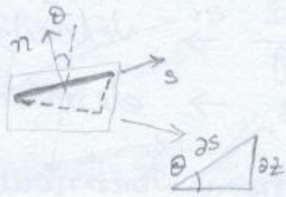
$$\therefore a_s = \frac{\partial V}{\partial s}\cdot\frac{\partial s}{\partial t} = V\frac{\partial V}{\partial s}$$

Therefore, the governing equation along the s-direction can be written as

$$-\frac{\partial P}{\partial s} - \rho g \sin\theta = \rho V\cdot\frac{\partial V}{\partial s}$$

from geometric analysis, we can write

$$\sin\theta = \frac{\partial z}{\partial s}$$



Geometric representation to find out '$\sin\theta$'

$$\therefore -\frac{\partial P}{\partial s} - \rho g\frac{\partial z}{\partial s} = \rho V\frac{\partial V}{\partial s} = \rho\frac{1}{2}\cdot\frac{\partial V^2}{\partial s}$$

$$\Rightarrow \frac{\partial P}{\partial s} + \frac{1}{2}\rho\frac{\partial V^2}{\partial s} + \rho g\frac{\partial z}{\partial s} = 0 \quad \text{(rearranging)}$$

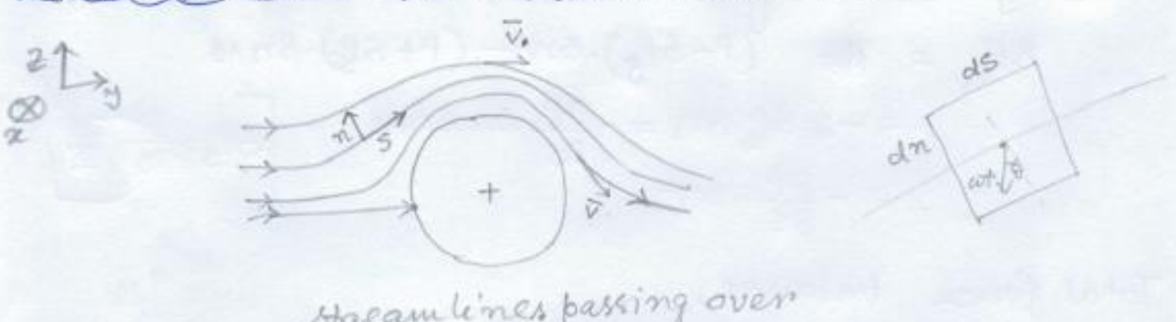
$$\Rightarrow \int\left[\frac{\partial P}{\partial s} + \frac{1}{2}\rho\frac{\partial V^2}{\partial s} + \rho g\frac{\partial z}{\partial s}\right] = C \quad \left[\text{integration along a streamline}\right]$$

So, for steady, incompressible, inviscid flow along a streamline, we get

$$\boxed{P + \frac{1}{2}\rho V^2 + \rho g z = C}$$ **(Bernoulli's equation)**

(Option2)

Bernoulli's equation.



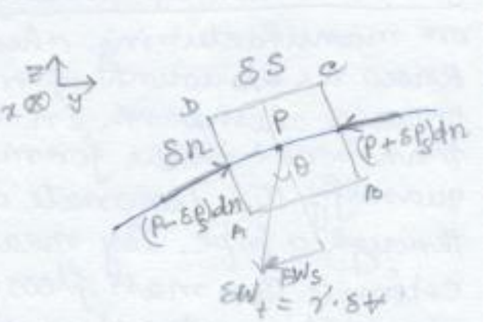
streamlines passing over an object.

Similar like previous derivation, we will assign the co-ordinates along the streamline and normal to the streamline as shown in the sketch. At every point, we get acceleration (two components)

$$a_s = \frac{dV}{dt} = \frac{\partial V}{\partial s}\cdot\frac{\partial s}{\partial t} = V\cdot\frac{\partial V}{\partial s}$$

and $a_n = \frac{V^2}{R}$

Now we take a streamline and consider a small differential volume as shown here. Length along 's' is δs and along 'n' is δn and normal to the page is of unit length. Weight of the element is acting downward with an angle 'θ' making with 'n' direction. Pressure at the centre is 'p'. Pressure force acting on plane AD is $(p-\delta p)\delta n \times 1$ and on plane BC is $(p+\delta p)\delta n \times 1$. Now, we need to compute the total force acting on the element.



differential volume element along a streamline of inviscid flow

$\boxed{\delta \forall = \delta s \cdot \delta n \times 1}$

Along s-direction,

$$\sum F_s = \delta m \, a_s = \delta m \cdot v \cdot \frac{\delta v}{\delta s} = \rho \delta \forall \cdot v \cdot \frac{dv}{ds} \quad \text{(Newton's second law)}$$

we also have another component due to weight.

$$\therefore \delta W_s = -\delta W_t \sin\theta = -\gamma \, \delta \forall \sin\theta$$

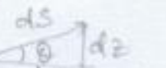
$\therefore$ force along s-direction due to pressure

$$\delta F_{p_s} = (p-\delta p_s)\cdot \delta n \times 1 - (p+\delta p_s)\cdot \delta n \times 1$$

$$= -2\,\delta p_s \, \delta n = -\frac{\partial p}{\partial s}\,\delta s \, \delta n \times 1 \qquad \left[\delta p_s = \frac{\partial p}{\partial s}\cdot\frac{\partial s}{2}\right]$$

$$= -\frac{\partial p}{\partial s}\cdot \delta \forall$$

Total force balance,



$\sin\theta = \frac{dz}{ds}$

$$\therefore \rho \cdot \delta \forall \cdot v \cdot \frac{\delta v}{\delta s} = -\gamma \cdot \delta \forall \cdot \sin\theta - \frac{\partial p}{\partial s}\cdot d\forall$$

$$\Rightarrow -\frac{\partial p}{\partial s} - \frac{1}{2}\rho \cdot \frac{d(v^2)}{ds} - \gamma \cdot \frac{dz}{ds} = 0$$

$$\Rightarrow dp + \frac{1}{2}\rho\, d(v^2) + \gamma \cdot dz = 0 \quad \text{(along a streamline)}$$

$$\Rightarrow \int \frac{dp}{\rho} + \frac{1}{2}v^2 + gz = \text{constant}$$

$$\Rightarrow \boxed{\frac{p}{\rho} + \frac{v^2}{2} + gz = C}$$ for incompressible fluid

Bernoulli's equation

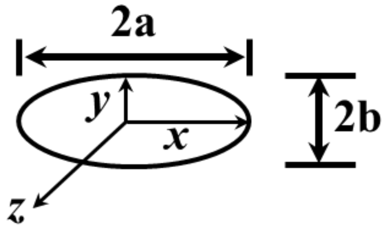
Q2. Q2. Consider a steady, laminar flow through a straight horizontal tube having the constant elliptical cross section given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The streamlines are all straight and parallel. Investigate the possibility of using an equation for the z-component of velocity of the form

$$w = K\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

as an exact solution to this problem. With this velocity distribution, what is the relationship between the pressure gradient along the tube and the volume flow rate through the tube?



Tube cross-section

(Marks: 7)

Answer:

Q1. Assumption:
1. $u=0$
2. $v=0$
3. No body force : $g_i=0$
4. Steady state : $\frac{\partial (\cdot)}{\partial t}=0$

continuity eqn:

$$\cancel{\frac{\partial u}{\partial x}}+\cancel{\frac{\partial v}{\partial y}}+\frac{\partial w}{\partial z}=0 \Rightarrow \frac{\partial w}{\partial z}=0$$

N.S eqn z-comp.

$$\rho\left\{\cancel{\frac{\partial w}{\partial t}}+\cancel{u\frac{\partial w}{\partial x}}+\cancel{v\frac{\partial w}{\partial y}}+\cancel{w\frac{\partial w}{\partial z}}\right\}=-\frac{\partial p}{\partial z}+\mu\left\{\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\cancel{\frac{\partial^2 w}{\partial z^2}}\right\}+\cancel{\rho g_z}$$

$$\frac{\partial p}{\partial z}=\mu\left\{\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right\}$$

The given velocity Profile: $w=A\left[1-\frac{x^2}{a^2}+\frac{y^2}{b^2}\right]$

BC : $w=0$ @ wall of Elliptical pipe

$$\Rightarrow \quad w=0 \quad @ \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Now put BC in velocity profile

$$0=A[1-1]=0 \quad \Rightarrow \text{u-satisfy the BC}$$

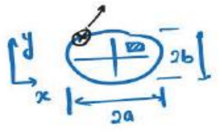
so to find out the $\frac{\partial p}{\partial z} \Rightarrow$ find out $\frac{\partial^2 w}{\partial x^2}$ & $\frac{\partial^2 w}{\partial y^2}$

$$\frac{\partial^2 w}{\partial x^2}=-\frac{2A}{a^2} \quad , \quad \frac{\partial^2 w}{\partial y^2}=-\frac{2A}{b^2}$$

$$\therefore \quad \frac{\partial p}{\partial z}=\mu\left[-\frac{2A}{a^2}-\frac{2A}{b^2}\right]=-2A\mu\left[\frac{1}{a^2}+\frac{1}{b^2}\right]$$

Now to find out the volume flow rate Q

$$Q=\int w\,dA \qquad A\rightarrow \text{Area (cross-section)}$$



$\Rightarrow dA=dx\,dy \Rightarrow Q=4\int_0^b\int_0^{a\sqrt{1-\frac{y^2}{b^2}}} w\,dx\,dy$

$y\rightarrow 0-b$
$x\rightarrow 0-a\sqrt{1-\frac{y^2}{b^2}}$ $\Big\}$ $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\Downarrow$$

$$Q=4A\int\int_0^{a\sqrt{1-\frac{y^2}{b^2}}}\left[1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right]dx\,dy$$

$$\Rightarrow \quad Q=4A\int_0^b\left[x-\frac{x^3}{3a^2}-\frac{y^2}{b^2}x\right]_0^{a\sqrt{1-y^2/b^2}}dy$$

$$=4A\int_0^b\left[a\sqrt{1-\frac{y^2}{b^2}}\left(1-\frac{y^2}{b^2}\right)-\frac{1}{3}a\sqrt{1-y^2/b^2}\left(1-\frac{y^2}{b^2}\right)\right]dy$$

$$\boxed{Q=\frac{8Aa}{3}\int_0^b\left(1-y^2/b^2\right)^{3/2}dy}$$ soln

$$Q=\frac{8Aa}{3}\times\frac{3b\pi}{16}=\frac{Aab\pi}{2}$$

Q3. A thin elastic wire is placed between rigid supports. A fluid flows past the wire, and it is desired to study the static deflection, δ, at the center of the wire due to the fluid drag. Assume that

$$\delta = f(l, d, \rho, \mu, V, E)$$

Where ***l*** is the wire length, ***d*** the wire diameter, ρ the fluid density, ***μ*** the fluid viscosity, ***V*** the fluid velocity, and ***E*** the modulus of elasticity of the wire material. Develop a suitable set of pi terms for this problem. **(Marks: 4)**

Answer:

Q3: π-terms: Given

$\delta = f(l, d, \rho, \mu, V, E)$

dimension of all

$\delta = L \qquad \rho = ML^{-3}$

$l = L \qquad \mu = ML^{-1}T^{-1}$

$d = L \qquad V = LT^{-1}$

$E = ML^{-1}T^{-2}$

1. No of physical parameters $\Rightarrow m = 7$
2. No of fundamental dimensions $\Rightarrow n = 3$

$\therefore$ No of π-terms $= m - n = 4$

use d, V, E as repeating variables

$\therefore \pi_1 = d^{x_1} V^{y_1} E^{z_1} \delta \rightarrow \pi_1 = \frac{\delta}{d}$

$\pi_2 = d^{x_2} V^{y_2} E^{z_2} l \rightarrow \pi_2 = l/d$

$\pi_3 = d^{x_3} V^{y_3} E^{z_3} \rho \rightarrow \pi_3 = \frac{\rho V^2}{E}$

$\pi_4 = d^{x_4} V^{y_4} E^{z_4} \mu \rightarrow \pi_4 = \left(\frac{\mu V}{dE}\right)$

π-Terms

Q4. A new blimp will move at 6 m/s in 20°C air, and we want to predict the drag force. Using a 1:13-scale model in water at 20°C and measuring a 2500-N drag force on the model, determine **(a)** the required water velocity, **(b)** the drag on the prototype blimp that will be required to propel it through the air. **(Marks: 2.5+2.5)**

Q4

(a) $Re_m = Re_p \Rightarrow \frac{D_m V_m \rho_m}{\mu_m} = \frac{D_p V_p \rho_p}{\mu_p} \Rightarrow V_m = \frac{D_p}{D_m} \times \frac{\rho_p}{\rho_m} \times \frac{\mu_m}{\mu_p} \times V_p$

$\therefore \boxed{V_m = \frac{D_p \rho_p \mu_m}{D_m \rho_m \mu_p} \times V_p} \Rightarrow \frac{D_p}{D_m} = 13 \Rightarrow V_m \simeq 5.2\ m/s$

(b) Drag:

$C_{Dm} = C_{Dp}$ $\qquad C_{D_i} = \frac{D_i}{\frac{1}{2}\rho_i V_i^2 \times A_{ref}}$ $\qquad A_{ref} \sim d^2$

$\Rightarrow \frac{D_m}{\cancel{\frac{1}{2}}\rho_m V_m^2 d_m^2} = \frac{D_p}{\cancel{\frac{1}{2}}\rho_p V_p^2 d_p^2}$

$D_p = \left(\frac{\rho_p}{\rho_m}\right)\left(\frac{V_p}{V_m}\right)^2\left(\frac{d_p}{d_m}\right)^2 \times D_m \simeq 678\ N$

Q5. The velocity profile for incompressible turbulent flow in a pipe of radius **R** is given $\mathbf{u(r) = u_{max}(1-r/R)^{1/7}}$. (a) Obtain an expression for the average velocity in the pipe, (b) Obtain an expression for the wall shear stress. **(Marks: 2+2)**

Answer:

Q5. Given: velocity profile:

$$u(r) = U_{Max}\left(1-\frac{r}{R}\right)^{1/7}$$

(a) Average velocity

$$\bar{u} = \frac{1}{A}\int u(r)\,dA \qquad A - \text{area}$$

for tube ⇒

$$\bar{u} = \frac{1}{\cancel{\pi} R^2}\int_0^R u(r)\,2\cancel{\pi} r\,dr$$

$$= \frac{2}{R^2}\int_0^R u_{max}\left[1-\frac{r}{R}\right]^{1/7} r\,dr$$

$$= \frac{2}{R^2}\left[(R+r)(7R+8r) - \frac{7}{120}\left(1-\frac{r}{R}\right)^{1/7}\right]_0^R u_{max}$$

$$= \frac{2\times 49\times R^2}{R^2\times 120}\times u_{max}$$

$$\boxed{\bar{u} = 0.8167\,u_{max}}$$

(b) wall shear stress

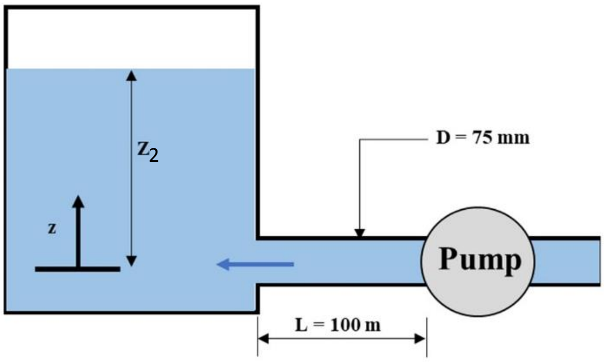
$$\tau = \mu\frac{\partial u}{\partial r}\bigg|_{r=R} \Rightarrow \frac{\partial u}{\partial r} = \frac{\partial}{\partial r}\left(1-\frac{r}{R}\right)^{1/7}$$

$$\frac{\partial u}{\partial r} = -\frac{1}{\left(1-\frac{r}{R}\right)^{6/7} R}\bigg|_{r=R} \rightarrow \infty$$

$$\boxed{\tau_w \rightarrow \infty}$$

Q6. Water is pumped at 0.01 m^3/s through a 75-mm-diameter smooth horizontal pipe, with L=100 m, into a constant-level reservoir of depth z_2 =10m as shown below. Find the pump pressure (in kPa), p_1, required to maintain the flow. Minor loss coefficient K=1.0, kinetic energy correction factor α=1.0, ρ=1000 kg/m^3, μ=10^{-3} kg/(m.s), g=9.8 m/s^2 and the friction coefficient, f, can be calculated from the following equation: $\frac{1}{\sqrt{f}} = -1.8\, log\left[\left(\frac{\frac{\varepsilon}{D}}{3.7}\right)^{1.11} + \frac{6.9}{Re}\right]$; where ε =0. Consider static head of the pump = 0. **(Marks: 5.0)**

($h_l = f\frac{L}{D}\frac{v^2}{2g}$ and $h_m = K\frac{v^2}{2g}$; 1kPa = 10^3 N/m^2)



Answer: Considering the losses in pipe flow, Bernoulli's equation can be written as,

$\frac{p_1}{\rho g} + \frac{\alpha {v_1}^2}{2g} + z_1 + H_p = \frac{p_2}{\rho g} + \frac{\alpha {v_2}^2}{2g} + z_2 + h_l + h_m$ (Point 1 is at pump and 2 is at the top of water body)

Where, H_p is pump head, h_l is major losses through pipe flow and h_m is minor losses.

Given: H_p =0, D= 75 mm =0.075 m, flow rate Q = 0.01 m^3/s, L =100m, ρ= 1000kg/m^3,

μ=10^{-3} kg/(m.s), g=9.8 m/s^2, α =1, K=1.

We need to find out p_1.

Comparing between point 1 and 2; z_1=0, z_2=10m, v_2=0, p_2=0 (gage), Δp= p_1-p_2=p_{pump}

as we know, $h_l = f\frac{L}{D}\frac{v^2}{2g}$ and $h_m = K\frac{v^2}{2g}$

Based on the data given, Bernoulli's equation is reduced to

$$\frac{p_1}{\rho g} + \frac{\alpha {v_1}^2}{2g} = z_2 + h_l + h_m$$

From the volumetric flow rate, v=v_1=Q/A, where A is cross-sectional area of the pipe.

A= $\pi D^2/4$

Hence, $v = \frac{4Q}{\pi D^2} = \frac{4\times 0.01}{\pi \times 0.075^2} = 2.26\, m/s$ and $Re = \frac{Dv\rho}{\mu} = \frac{0.075\times 2.26\times 1000}{10{-}3} = 1.7\times 10^5$

So, $\frac{1}{\sqrt{f}} = -1.8\, log\left[\left(\frac{\frac{\varepsilon}{D}}{3.7}\right)^{1.11} + \frac{6.9}{Re}\right] = -1.8\log\left(\frac{6.9}{Re}\right) = -1.8\log(4.06\times 10^{-5}) = 7.9$

And f = 0.016

So, $p_{pump} = p_1 = (z_2 + h_l)\rho g = \left(10 + 0.016 \times \frac{100}{0.075} \times \frac{2.26^2}{2\times 9.8}\right) \times 10^3 \times 9.8 = 1.52 \times 10^5$ N/m^2 (gage)

Finally, $\boldsymbol{p_{pump} = 152\ kPa\ (gage)}$............(answer)

Q7. In tests of a protective material, one bullet hits it at a speed of 0.55 km/s; Calculate the speed of sound and the Mach number of the bullet if the ambient temperature is 280 K and the gas constant of air = 287 J/(kg.K) . The specific heat ratio of air =1.4. **(Marks: 2+1)**

Answer: Speed of sound is calculated by the following equation: $a = \sqrt{kRT}$ where 'a' is speed of sound, 'R' gas constant of air and 'T' is absolute temperature.

Given: k=1.4, R= 287 J/(kg.K) and T= 280 K

So, $\boldsymbol{a = \sqrt{1.4 \times 287 \times 280}\ \frac{m}{s} = 335.42\ m/s}$ ....(answer)

And the Mach number; $Ma = \frac{V}{a}$ where 'V' speed of the object and 'a' is the speed of sound.

So, $\boldsymbol{Ma = \frac{550}{335.42} = 1.64}$ ....(answer)