6) Given the wavefunction for a particle in a 1D box (PIB) of length $\boldsymbol{l}$, **(5x1= 5 marks)**

$$\psi(x) = \sqrt{\frac{2}{l}}\left[\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l}\right]$$

Realize that the wavefunction is superposition of $n = 1{,}2\ and\ 3$ states.

a) Normalize the wavefunction

$$\int_0^1 \boldsymbol{\psi^*\psi d\tau = 1}$$

$$\Rightarrow \int_0^l \sqrt{\frac{2}{l}}\left[\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l}\right] \times \sqrt{\frac{2}{l}}\left[\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l}\right] dx$$

$$= \int_0^l \frac{2}{l}\sin\frac{\pi x}{l}\sin\frac{\pi x}{l}dx + \frac{1}{4}\int_0^l \frac{2}{l}\sin\frac{2\pi x}{l}\sin\frac{2\pi x}{l}dx + \frac{1}{9}\int_0^l \frac{2}{l}\sin\frac{3\pi x}{l}\sin\frac{3\pi x}{l}dx$$

(since the eigenfunctions are orthonormal)

$$= 1 + \frac{1}{4} + \frac{1}{9} = \frac{36+9+4}{36} = \frac{49}{36}$$

$$\Rightarrow \sqrt{\frac{36}{49}}\left(\sqrt{\frac{2}{l}}\left[\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l}\right]\right)$$

is the normalized wavefunction.

b) Use the normalized wavefunction to:

i. Determine the probability of finding the particle in the first excited state of the PIB

Probability is $|C_i|^2$ **when** $\psi = c_1\phi_1 + c_2\phi_2 + \cdots$ **The first E.S. is when** $n = 2$**.**

$$c_2 = \frac{1}{2}\sqrt{\frac{36}{49}} \Rightarrow |C_i|^2 = \frac{1}{4}\times\frac{36}{49} = \frac{9}{49}$$

ii. Determine the average energy of the particle

$$<E> = \int_0^l \psi^* \hat{H}\psi dx = \int_0^l (c_1\phi_1 + c_2\phi_2 + c_3\phi_3)^* \hat{H}(c_1\phi_1 + c_2\phi_2 + c_3\phi_3)dx$$

$$= \int_0^l (c_1\phi_1 + c_2\phi_2 + c_3\phi_3)^*(c_1E_1\phi_1 + c_2E_2\phi_2 + c_3E_3\phi_3)dx$$

$$= c_1^2E_1 + c_2^2E_2 + c_3^2E_3 \text{ (orthonormal eqigenfunctions)}$$

$$= \frac{36}{49}\left(1 \times 1 + \frac{1}{4} \times 4 + \frac{1}{9} \times 9\right)\frac{h^2}{8ml^2} = \frac{108}{49}\frac{h^2}{8ml^2}$$

c) Is the wavefunction orthogonal to the second excited state of the PIB? Show.

No, since it has the second excited state.

$$\int_0^l \sqrt{\frac{2}{l}} \sin\frac{3\pi x}{l}\left\{\sqrt{\frac{36}{49}}\left(\sqrt{\frac{2}{l}}\left[\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l}\right]\right)\right\}dx$$

$$= 0 + 0 + \frac{1}{3} \times \sqrt{\frac{36}{49}}$$

d) Estimate the uncertainty in the momentum of the particle.

$$\sigma_{p_x} = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\langle 2mE\rangle - 0} = \sqrt{\frac{108}{49}\frac{h^2}{4l^2}}$$

(from $E = p^2/2m$ and the fact that the average value of momentum of a PIB moving in both directions is zero. Value of E is from part ii.)