Entry number:

There are 7 questions for a total of 30 points.
(*Please make sure that you write the question numbers correctly*)

1. Answer the following questions.
   (a) (1 point) Recall the Longest Increasing Subsequence problem discussed in class. Consider the sequence of numbers in array $A = [14, 8, 2, 7, 4, 10, 5, 0, 1, 9, 6, 13, 3, 11, 12, 15]$. As in the class discussion, let $L(i)$ denote the length of the longest increasing subsequence of $A[1...n]$ that ends with $A[i]$. Fill the table for $L[1...16]$ as shown below.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$L[i]$																

   (b) (1 point) n balls are thrown randomly into n distinguishable bins. What is the probability that the first bin has exactly k balls? Give a concise expression in terms of k and n. Show how you arrived at your solution.

   (c) (1 point) State true or false with reasons: Let A, B, and C be sets. If for all x,
$$x \in A \rightarrow (x \in B \rightarrow x \in C),$$
   then $A \cap B \subseteq C$.

   (d) (2 points) In how many ways can you distribute n indistinguishable apples, one orange, and one banana to k children such that each child gets at least one fruit? Give reasons. Assume that $n > k + 2$.

2. (3 points) State true or false with reasons: For every $n > 0$, $2903^n - 803^n - 464^n + 261^n$ is divisible by 1897.

3. The following information is available about a random variable X: (i) $0 \leq X \leq 100$ and (ii) $\mathbf{E}[X] = 70$.
   (a) (1 point) What is the maximum value that $\mathbf{Pr}[X = 100]$ could take? (*The maximum is over all possibilities for X that satisfy condition (i) and (ii) above*). Briefly explain your answer.
   (b) (1 point) Suppose we change the condition (i) to $20 \leq X \leq 100$ (condition (ii) remains same). What is the maximum value $\mathbf{Pr}[X = 100]$ could take? Briefly explain your answer.

4. Consider an undirected graph $G = (V, E)$ with n vertices and m edges (there are no self loops or multi edges in G). Let us randomly partition the vertices of the graph into two sets A, B (i.e., for any vertex v it is in A with probability 1/2). Let X be the random variable denoting the number of edges that do not have both their endpoints in the same partition? Answer the following giving reasons for each part.
   (a) (1 point) What is the value of $\mathbf{E}[X]$?
   (b) (2 points) What is the value of $\mathbf{Var}[X]$?
   (c) (½ point) Apply Chebychev's inequality to give an upper bound on the probability that the number of edges that do not have both endpoints in the same partition is at most $m/4$.

5. (5 points) Consider the following recursive program:

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Rec(i, j)
  - if (i = 0 or j = 0)return(1)
  - return(Max(Rec(i - 1, j), Rec(i, j - 1)))
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Here Max(.,.) is a subroutine that returns the maximum of two input numbers. Let $T(n)$ denote the number of times the Max subroutine is called during the execution of Rec(n, n). Give a concise expression for $T(n)$ in terms of $n > 0$. Argue correctness of your expression.

6. A Quaternary string is a string of numbers from the set $\{0, 1, 2, 3\}$ (*similar to binary strings that are strings of just 0's and 1's*). Let $T(n)$ denote the number of Quaternary strings of length n that have either two consecutive 0's or two consecutive 1's (*for example,* 002111 *and* 01123011 *are such strings*). So, $T(1) = 0$ and $T(2) = 2$. Answer the following questions giving explanations.
   (a) (3 points) Write a recurrence relation for $T(n)$.
   (*Hint: You can write $T(n)$ in terms of $T(n-1)$ and $T(n-2)$*).
   (b) (3 points) Give an exact expression for $T(n)$ as a function of n.
   (c) (1 point) Give the value of $T(6)$.

7. One can obtain stronger tail bounds than the Markov's inequality for random variables that satisfy certain conditions. In this question, we consider one such example. Let $X_1, ..., X_n$ be independent 0/1 random variables and let $p_i = \mathbf{E}[X_i]$ for $i = 1, ..., n$. Let $X = X_1 + ... + X_n$ and let $\mu = \mathbf{E}[X]$.
   (a) (4 points) Show that for any real $\beta > 1$:
   $$\mathbf{Pr}[X \geq \beta \cdot \mu] \leq e^{-g(\beta)\cdot\mu}$$
   where the function $g(.)$ is defined as $g(\beta) = \beta \ln \beta + 1 - \beta$.

   (*Hint: Consider the random variable $Y = e^{\lambda X}$ and apply Markov's inequality choosing λ carefully to optimise the probability bound. You may use the inequality $(1 + z) \leq e^z$ for any real z.*)
   (b) (½ point) Use part (a) to give upper-bound on the probability of getting at least $3n/4$ heads when an unbiased coin is tossed n times.