COL202: Discrete Mathematical Structures. I semester, 2020-21.
Major exam
6 January 2021
Maximum Marks: 42

Important: You are allowed to use only two A4 sides of a page in a reasonable sized handwriting for each question (including all subparts). Write your name on each side of a page submitted (all 10 pages of your pdf). **If we can't understand what you have written we don't give you any marks**.

Problem 1 (Total 8 = 3 + 2 + 3 marks)
Some problems in logic.

Problem 1.1 (3 marks)
Let $D = \{a, b, c, d\}$ be a set and $P, Q, R : D \to \{T, F\}$ be three predicates such that only the following are true: $P(a), Q(a), Q(b), P(d), R(d)$. Prove or disprove:

$$\forall x \in D : (P(x) \Rightarrow Q(x)) \Rightarrow R(x).$$

Problem 1.2 (2 marks)
Let $P[x]$ be the set of polynomials in x with real coefficients. Given $f(x) \in P[x]$ write a predicate (in logic notation) that is true if and only if $f(x)$ has real roots.

Problem 1.3 (3 marks)
Let $f'(x)$ denote the derivative of a polynomial $f(x) \in P[x]$. Write the following theorem as a logical statement (in logic notation): If $f(x)$ has real roots then its derivative is either a scalar or has real roots.

Problem 2 (Total 7 = 4 + 3 marks)
Given a tree T, $\mathcal{T}$ is the set of all subtrees of T, i.e., all connected subgraphs of T. We denote by $\subseteq_T$ the subgraph relation on $\mathcal{T}$.

Problem 2.1 (4 marks)
Prove that $(\mathcal{T}, \subseteq_T)$ is a poset.

Problem 2.2 (3 marks)
Is $(\mathcal{T}, \subseteq_T)$ a lattice?

Problem 3 (Total 8 = 3 + 2 + 3 marks)
Given a permutation $\pi : [n] \to [n]$ of the first n natural numbers, we say that $j \in [n]$ is a *winner* in π if $\pi(j) > \pi(i)$ for all $1 \leq i \leq j$. We assume that 1 is a winner in every permutation.

Problem 3.1 (3 marks)
Prove that for each $k \in [n]$, the number of permutations of $[n]$ with exactly k winners is

$$[x^k] \prod_{i=0}^{n-1} (i + x).$$

Problem 3.2 (2 marks)
We choose a permutation of $[n]$ uniformly at random from all possible permutations. Use the result of Problem 3.1 to find the expected number of winners in the permutation. If you do not use Problem 3.1 you can "expect" to get a 0 in this part.

Problem 3.3 (3 marks)
Now solve Problem 3.2 again using the concepts of indicator random variables and linearity of expectation. Again, if you do not use indicator variables along with linearity of expectations here you will receive a 0.

Problem 4 (Total 10 = 3 + 7 marks)
We are given the probability space $(\{0,1\}^k, \mathcal{F} = 2^{\{0,1\}^k})$, i.e., each outcome is a bit string of length k and the σ-algebra is the power set of the outcome space. We define random variable $X_i(\omega)$ as the ith coordinate of the outcome ω, $1 \leq i \leq k$.

Problem 4.1 (3 marks)
Suppose we choose ω uniformly at random from $\{0,1\}^k$. Show that the collection of random variables $\{X_i : 1 \leq i \leq k\}$ is mutually independent.

Problem 4.2 (7 marks)
Note that $(\{0,1\}^k, \leq_k)$ is a partially ordered set which is the k-way product of the partially ordered set $(\{0,1\}, \leq)$ where $\leq$ is the usual order on integers, i.e., for $a, b \in \{0,1\}^k$ if we say $a \leq_k b$ whenever $a_i \leq b_i$ then $\leq_k$ is a partial order on $\{0,1\}^k$. Now, consider two events, $A, B \in \mathcal{F}$, which are up sets of the partially ordered set $(\{0,1\}^k \leq_k)$ ($S \subset X$ is an *up set* of a poset $(X, \preceq)$ if $x \in S$ implies that $y \in S$ for every y such that $x \preceq y$). Prove by induction on k that these two events are *positively correlated*, i.e.,

$$\mathrm{P}\{A \cap B\} \geq \mathrm{P}\{A\}\mathrm{P}\{B\}.$$

Problem 5 (Total 9 = 3 + 6 marks)
Given a set of vertices V with $|V| = n + 1$ and a $p \in (0,1)$, we construct a random simple graph $G_p = (V, E_p)$ as follows: Each pair $(u, v) \in V \times V$ such that $u \neq v$ is placed in E_p with probability p independent of all other pairs.

Problem 5.1 (4 marks)
Let A be the event that G_p has a clique of size k (where $2 \leq k \leq n+1$). Argue that $\mathrm{P}\{A\} \leq \binom{n+1}{k} p^{\binom{k}{2}}$.

Problem 5.2 (6 marks)
If $d_{\max}$ is the maximum degree of $G_{\frac{1}{n}}$ then show that there is a constant $c > 0$ such that

$$\mathrm{P}\{d_{\max} > c \log n\} \leq \frac{1}{n}.$$

You must also mention the value of c for which this is true.