COL202: Discrete Mathematical Structures. I semester, 2017-18.
Minor I
27 August 2017
Maximum Marks: 15 + 3 bonus

Name	**I**	**Ent. No.**

Important: Keep your answers within the boxes prescribed for each question. Anything written outside the box will be treated as rough work. Solve the problem first on the separate rough sheets provided then copy carefully into the printed sheet. **Rough sheets will not be collected.** Starred (*) problems may take more time. The marks associated with them are bonus marks and so these problems carry fewer marks. This does *not* mean they are easier than the unstarred problems with more marks.

Problem 1 (2+1*+3+2+2+1*+1*=9+3 bonus marks)
A plane has seats 1 to n. Outside the plane are passengers standing in line according to their seat number with passenger 1 standing at the front and passenger n at the end. The passengers are allowed to enter the plane one at a time in ascending order of their seat number. Passenger 1 is a VIP and is although he is supposed to sit in seat 1 he is allowed to choose any seat (i.e. he does *not* have to sit on seat 1 although he can if he wants to). All other passengers are non-VIPs and follow the rule

> For $i \geq 2$, if seat i is available when passenger i enters the plane she sits on seat i otherwise she is allowed to choose any of the seats that are not already filled.

Answer the questions that follow about the seating orders that are possible in this setting.

Problem 1.1 (2 marks)
Give an example that shows that not all $n!$ seating orders are possible. Set $n = 5$ for the example you give. To get > 0 marks you must give a one line argument as to why your seating order is not possible.

Problem 1.2 (1 mark)*
Prove that the number of seating orders in which passenger n sits in seat n are equal to the number of seating orders in which passenger n sits in seat 1. There are multiple ways of doing this but the way we want you to approach this involves the bijection principle. If S_n^i is the set of seatings where passenger n sits in seat i ($i = 1, n$) find a bijection between S_n^1 and S_n^n to give this proof. *Any attempt to solve this part by calculating the sizes of S_n^1 or S_n^n will get 0 marks even if it is correct.*

Problem 1.3 (3 marks)
If a_n is the number of seating orders for n passengers, find a recurrence for a_n. You may assume $a_0 = 0$ and $a_1 = 1$. Recall that a plain answer with no reasoning will get you 0 marks even if correct.

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Problem 1.4 (2 marks)
Solve the recurrence found in Subproblem 1.3 by inspecting its terms, guessing the answer and proving the answer is correct by induction (the so-called "plug and chug" method.).

Problem 1.5 (2 marks)
Use the recurrence found in Subproblem 1.3 to find the generating function $A(x)$ for the sequence $\{a_n\}_{n\geq 0}$. Check that your generating function is correct by comparing the coefficient of x^n with the answer you obtained for Subproblem 1.4.

Problem 1.6 (1 mark)*

There is a way of finding the number of seating orders without recurrences or generating functions that uses the following identity involving the binomial coefficients

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Describe this argument.

Problem 1.7 (1 mark)*

Extend the argument made in Subproblem 1.6 to solve Subproblem 1.2 by determining the sizes of S_n^1 and S_n^n in terms of n and showing that they are equal.

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Problem 2 (2 marks)
Given a circular table with 12 seats placements, each of them marked with a name, we find that when the 12 people invited sit down they are all sitting at a place with someone else's name on it. The host suggests that instead of people getting up and reseating themselves, he will just rotate the table. Prove that there is a position obtained by rotation in which at least 2 people are sitting in front of the place with their own name on it.

Problem 3 (1+2 = 3 marks)
In this problem we will create a bijection between the positive real numbers and the entire set of real numbers. We will do it in two steps.

Problem 3.1 (1 mark)
Define a bijection between the set $(0, 1]$ and $(0, 2]$ (where $(a, b]$ denotes all real numbers between a and b, excluding a and including b.). Note you are asked to create a bijection, not show by any other means that the two sets have the same cardinality.

Problem 3.2 (2 marks)
Use the bijection created in Problem 3.1 to create a bijection between the set of reals number $\mathbb{R}$ and the set $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$. If you were not able to solve Problem 3.1, you can still assume a bijection exists between $(0, 1]$ and $(0, 2]$.