Name: *Entry number:* *Group number:*

ELL205 Signals and Systems: Minor 2

Time: 60 mins *Maximum points: 25*

- *Write your answers in the space provided in the question paper.*
- *Additional sheets would be provided only for rough work.*
- *Justify your answers clearly; answers without justification will receive no credit.*

1. Signal $x(t)$ is periodic with period 10 and given by $\cos\left(\frac{3\pi t}{10}\right)$ for $t \in [-5, 5)$.

   (a) Find the Fourier series coefficients a_k of $x(t)$. . . . . **(2 points)**

   (b) The signal $x(t)$ is input to a system with frequency response:

   $$H(\omega) = \begin{cases} 1 & |\omega| \geq \frac{3\pi}{10} \\ 0 & |\omega| < \frac{3\pi}{10} \end{cases}$$

   Find the output signal $y(t)$ for $t \in [-5, 5)$. . . . . **(2 points)**

Solution:

(a) The Fourier series coefficients are given by:

$$a_k = \frac{1}{T}\int_T x(t)\exp(-j\frac{2\pi}{T}kt)dt = \frac{1}{10}\int_{-5}^{5}\frac{1}{2}\left(e^{j\frac{3\pi}{10}t} + e^{-j\frac{3\pi}{10}t}\right)e^{-j\frac{\pi}{5}kt}dt$$

. . . . **(0.5 points)**

$$a_k = \frac{1}{20}\int_{-5}^{5} e^{j\left(\frac{3-2k}{10}\right)\pi t}dt + \frac{1}{20}\int_{-5}^{5} e^{-j\left(\frac{3+2k}{10}\right)\pi t}dt$$

$$= \frac{1}{j2\pi(3-2k)}\left[e^{j(3-2k)\frac{\pi}{2}} - e^{-j(3-2k)\frac{\pi}{2}}\right] - \frac{1}{j2\pi(3+2k)}\left[e^{-j(3+2k)\frac{\pi}{2}} - e^{j(3+2k)\frac{\pi}{2}}\right]$$

. . . . **(0.5 points)**

$$a_k = \frac{\sin\left((3-2k)\frac{\pi}{2}\right)}{\pi(3-2k)} + \frac{\sin\left((3+2k)\frac{\pi}{2}\right)}{\pi(3+2k)}.$$

. . . . **(1 points)**

(b) Since $\omega_0 = \frac{2\pi}{10}$, the harmonic components of the input with $k = 0, \pm 1$ will be eliminated by the system. Thus:

$$y(t) = x(t) - a_{-1}e^{-j\omega_0 t} - a_0 - a_{-1}e^{j\omega_0 t}$$

. . . . **(1 points)**

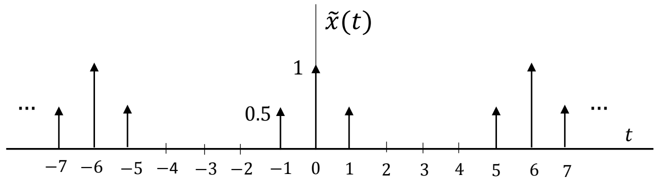
$a_0 = -\frac{2}{3\pi}$ and $a_1 = a_{-1} = \frac{6}{5\pi}$. . . . . **(0.5 points)**

$$y(t) = \cos\left(\frac{3\pi t}{10}\right) - \frac{6}{5\pi}\left[e^{-j\frac{\pi}{5}t} + e^{j\frac{\pi}{5}t}\right] + \frac{2}{3\pi}$$

$$y(t) = \cos\left(\frac{3\pi t}{10}\right) - \frac{12}{5\pi}\cos\left(\frac{\pi t}{5}\right) + \frac{2}{3\pi}$$

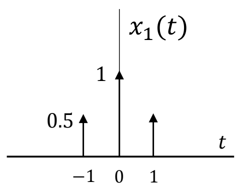
. . . . **(0.5 points)**

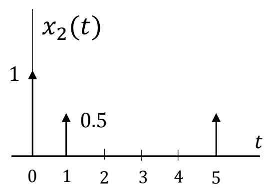
2. Consider the periodic signal $\tilde{x}(t)$ shown below:



(a) Determine the Fourier series representation for $\tilde{x}(t)$. . . . . **(2 points)**

(b) Determine the Fourier transforms of the signals $x_1(t)$ and $x_2(t)$ shown below: . . . . **(2 points)**





(c) Express the Fourier series coefficients determined in part (a) in terms of the Fourier transforms of $x_1(t)$ and $x_2(t)$. . . . . **(1 points)**

(d) Comment on the similarities/dissimilarities of the Fourier transforms of $x_1(t)$ and $x_2(t)$. . . . . **(1 points)**

Solution:

(a) $\tilde{x}(t)$ is periodic with period $T = 6$, so its Fourier series coefficients are given by: . . . . **(0.5 points)**

$$a_k = \frac{1}{T}\int_T \tilde{x}(t)e^{-jk\frac{2\pi}{T}t}dt = \frac{1}{6}\int_{-3}^{3}\left[\frac{1}{2}\delta(t+1) + \delta(t) + \frac{1}{2}\delta(t-1)\right]e^{-jk\frac{2\pi}{6}t}dt$$
$$= \frac{1}{6}\left[\frac{1}{2}e^{j\frac{\pi k}{3}} + 1 + \frac{1}{2}e^{-j\frac{\pi k}{3}}\right] = \frac{1}{6}\left(1 + \cos\left(\frac{\pi k}{3}\right)\right)$$

. . . . **(1 points)**

Thus, $\tilde{x}(t) = \sum_{k=-\infty}^{\infty}\frac{1}{6}\left(1+\cos\left(\frac{\pi k}{3}\right)\right)e^{jk\frac{\pi}{3}t}$. . . . . **(0.5 points)**

(b)

$$X_1(\omega) = \int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty}\left[\frac{1}{2}\delta(t+1) + \delta(t) + \frac{1}{2}\delta(t-1)\right]e^{-j\omega t}dt$$
$$X_1(\omega) = \left[\frac{1}{2}e^{j\omega} + 1 + \frac{1}{2}e^{-j\omega}\right] = 1 + \cos\omega$$

. . . . **(1 points)**

$$X_2(\omega) = \int_{-\infty}^{\infty}\left[\delta(t) + \frac{1}{2}\delta(t-1) + \frac{1}{2}\delta(t-5)\right]e^{-j\omega t}dt = \left[1 + \frac{1}{2}e^{-j\omega} + \frac{1}{2}e^{-j5\omega}\right]$$

. . . . **(1 points)**

(c) It is seen that by periodically repeating $x_1(t)$ or $x_2(t)$ with period $T = 6$ we obtain $\tilde{x}(t)$. Thus, the Fourier series coefficients of $\tilde{x}(t)$ are the scaled samples of $X_1(\omega)$ or $X_2(\omega)$ at the harmonically related frequencies $k\omega_0 = \frac{2\pi k}{6}$. .... **(0.5 points)**

$$X_1(\omega)|_{\omega=\frac{2\pi k}{6}} = 1 + \cos\left(\frac{2\pi k}{6}\right) = 6a_k$$

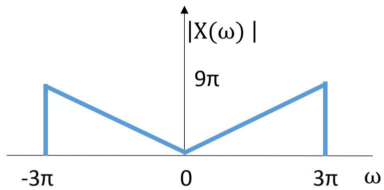
Similarly:

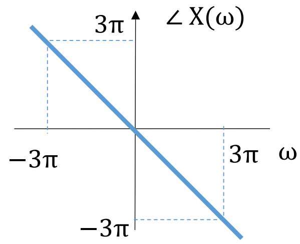
$$\begin{aligned} X_2(\omega)|_{\omega=\frac{2\pi k}{6}} &= 1 + \frac{1}{2}e^{-j\frac{2\pi k}{6}} + \frac{1}{2}e^{-j\frac{10\pi k}{6}} = 1 + \frac{1}{2}e^{-j\frac{2\pi k}{6}} + \frac{1}{2}e^{j\frac{2\pi k}{6}} \\ &= 1 + \cos\left(\frac{2\pi k}{6}\right) = 6a_k \end{aligned}$$

.... **(0.5 points)**

(d) It is noted that although $X_1(\omega) \neq X_2(\omega)$, but they are equal at $\omega = \frac{2\pi k}{6}$. .... **(1 points)**

3. Find $x(t)$ whose Fourier transform $X(\omega)$ has the magnitude and phase as shown below. . . . . **(5 points)**





Solution:

$$X(\omega) = |X(\omega)|\, e^{j\angle X(\omega)} = \begin{cases} 3|\omega|e^{-j\omega} & \text{for } |\omega| \le 3\pi \\ 0 & \text{otherwise} \end{cases}$$

. . . . **(0.5 points)**

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\left[\int_{-3\pi}^{0} -3\omega e^{-j\omega}e^{j\omega t}d\omega + \int_{0}^{3\pi} 3\omega e^{-j\omega}e^{j\omega t}d\omega\right] \\ &= \frac{1}{2\pi}\left[\int_{0}^{3\pi} 3\omega e^{j\omega}e^{-j\omega t}d\omega + \int_{0}^{3\pi} 3\omega e^{-j\omega}e^{j\omega t}d\omega\right] \\ &= \frac{3}{\pi}\int_{0}^{3\pi} \omega\cos\left(\omega(t-1)\right)d\omega = \frac{3}{\pi}\left[\frac{\omega\sin\left(\omega(t-1)\right)}{(t-1)} + \frac{\cos\left(\omega(t-1)\right)}{(t-1)^2}\right]_{0}^{3\pi} \end{aligned}$$

. . . . **(3.5 points)**

$$\begin{aligned} x(t) &= \frac{3}{\pi}\left[\frac{3\pi\sin\left(3\pi(t-1)\right)}{(t-1)} + \frac{\cos\left(3\pi(t-1)\right)}{(t-1)^2} - \frac{1}{(t-1)^2}\right] \\ &= \frac{-3}{\pi}\left[\frac{3\pi\sin\left(3\pi t\right)}{(t-1)} + \frac{\cos\left(3\pi t\right)}{(t-1)^2} + \frac{1}{(t-1)^2}\right] \end{aligned}$$

. . . . **(1 points)**

4. The output $y(t)$ of an LTI system S for a given input $x(t)$ is:

$$y(t) = \left(x(t)\sin^2(t)\right) * \frac{\cos t}{\pi\left(t - \frac{\pi}{2}\right)}.$$

Assuming $x(t)$ is real and $X(\omega) = 0$ for $|\omega| \geq 1$, find the impulse response of the system S. .... **(5 points)**

Solution: Note that $\frac{\cos t}{\pi\left(t-\frac{\pi}{2}\right)} = \frac{-\sin\left(t-\frac{\pi}{2}\right)}{\pi\left(t-\frac{\pi}{2}\right)}$. Consider the FT pairs below:

$$\begin{aligned}
\text{rect}\left(\frac{t}{T}\right) &\leftrightarrow \frac{2\sin\left(\frac{\omega T}{2}\right)}{\omega} \\
\frac{2\sin\left(\frac{tT}{2}\right)}{t} &\leftrightarrow 2\pi\text{rect}\left(\frac{\omega}{T}\right) \text{ (by duality)} \\
\frac{\sin t}{t} &\leftrightarrow \pi\text{rect}\left(\frac{\omega}{2}\right) \text{ (assuming } T = 2\text{)} \\
\frac{\sin\left(t - \frac{\pi}{2}\right)}{\left(t - \frac{\pi}{2}\right)} &\leftrightarrow \pi\text{rect}\left(\frac{\omega}{2}\right) e^{-j\frac{\pi\omega}{2}} \\
\frac{-\sin\left(t - \frac{\pi}{2}\right)}{\pi\left(t - \frac{\pi}{2}\right)} &\leftrightarrow -\text{rect}\left(\frac{\omega}{2}\right) e^{-j\frac{\pi\omega}{2}}
\end{aligned}$$

.... **(2 points)**

$$\begin{aligned}
\sin^2(t) = \frac{1-\cos 2t}{2} &\leftrightarrow \frac{1}{2}\left[2\pi\delta(\omega) - \pi\delta(\omega-2) - \pi\delta(\omega+2)\right] \\
x(t)\sin^2(t) &\leftrightarrow \frac{1}{2\pi}\left(X(\omega) * \frac{1}{2}\left[2\pi\delta(\omega) - \pi\delta(\omega-2) - \pi\delta(\omega+2)\right]\right) \\
x(t)\sin^2(t) &\leftrightarrow \left(\frac{1}{2}X(\omega) - \frac{1}{4}X(\omega-2) - \frac{1}{4}X(\omega+2)\right)
\end{aligned}$$

.... **(2 points)**

Thus, $Y(\omega) = -\text{rect}\left(\frac{\omega}{2}\right) e^{-j\frac{\pi\omega}{2}} \cdot \left[\frac{1}{2}X(\omega) - \frac{1}{4}X(\omega-2) - \frac{1}{4}X(\omega+2)\right]$.

However, both $X(\omega)$ and $\text{rect}\left(\frac{\omega}{2}\right)$ are zero for $|\omega| \geq 1$.

Thus, $Y(\omega) = -\frac{X(\omega)}{2}e^{-j\frac{\pi\omega}{2}}$ .... **(0.5 points)**

$\implies H(\omega) = -\frac{1}{2}e^{-j\frac{\pi\omega}{2}}$ or $h(t) = -\frac{1}{2}\delta\left(t - \frac{\pi}{2}\right)$. .... **(0.5 points)**

5. Indicate which of the following statements are true/false with justification.

 (a) $H(0)$ is the maximum value of the frequency response if $h(t)$ is real and positive. . . . . **(1 points)**

 (b) For the signal $\text{rect}(\frac{t}{T}) = \begin{cases} 1, & |t| < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$, the equivalent rectangular bandwidth (obtained by equating areas, in Hz) is $\frac{1}{T}$. . . . . . . **(2 points)**

 (c) The set represented by $\left\{ \frac{\sin\left(\frac{\omega T}{2} - k\pi\right)}{\left(\frac{\omega T}{2} - k\pi\right)} e^{-jk\omega T} \right\}$ (indexed by k) is orthogonal. . . . . **(2 points)**

Solution:

(a)

$$|H(\omega)| = \left| \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \right| \leq \int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} h(t) dt = H(0)$$

Thus, $|H(\omega)| \leq H(0)$ and the given statement is true. . . **(1 points)**

(b) The following are a Fourier transform pair:

$$x(t) = \text{rect}\left(\frac{t}{T}\right) \leftrightarrow X(\omega) = \frac{2\sin\left(\frac{\omega T}{2}\right)}{\omega}$$

Area of $X(\omega) = \int_{-\infty}^{\infty} X(\omega) d\omega = 2\pi x(0)$ . . . . **(1 points)**

Area of the equivalent rectangle $= 2WX(0)$, where W is the single-sided bandwidth (in radians/s) and $X(0) = T$.

Equating these two areas: $W = \frac{2\pi x(0)}{2X(0)} = \frac{\pi}{T}$ (in rad/s) $= \frac{1}{2T}$ (in Hz).
. . . . **(0.5 points)**

Thus, given statement is false. . . . . **(0.5 points)**

(c) We have the following Fourier transform pairs:

$$\text{rect}\left(\frac{t}{T}\right) \leftrightarrow \frac{2\sin\left(\frac{\omega T}{2}\right)}{\omega}$$

$$e^{jk\omega_0 t}\text{rect}\left(\frac{t}{T}\right) \leftrightarrow \frac{2\sin\left(\frac{(\omega - k\omega_0)T}{2}\right)}{(\omega - k\omega_0)} \text{ where } \omega_0 = \frac{2\pi}{T}$$

$$e^{jk\omega_0(t-kT)}\text{rect}\left(\frac{t-kT}{T}\right) \leftrightarrow \frac{2\sin\left(\frac{(\omega - k\omega_0)T}{2}\right)}{(\omega - k\omega_0)} e^{-jk\omega T}$$

$$e^{j\frac{2\pi k}{T}(t-kT)}\text{rect}\left(\frac{t-kT}{T}\right) \leftrightarrow \frac{T\sin\left(\frac{\omega T}{2} - k\pi\right)}{\left(\frac{\omega T}{2} - k\pi\right)} e^{-jk\omega T}$$

. . . . **(1.5 points)**

It can be shown that for two signals $u(t)$ and $v(t)$:

$$\int_{-\infty}^{\infty} u(t) v^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(\omega) V^*(\omega) d\omega$$

Thus, if two signals are orthogonal in the time-domain they will also be orthogonal in the frequency-domain, and vice versa.

Note that $\text{rect}\left(\frac{t-kT}{T}\right)$ is a set of rectangular pulses of width T that are shifted by integral multiples of T, and thus are non-overlapping and orthogonal. The last FT pair above is orthogonal in time-domain and hence in frequency-domain as well. Given statement is true.
. . . . **(0.5 points)**