ELL333 MAJOR TEST MARKS = 35 TIME = 2 HOURS

1. Consider the system $\ddot{y} - a^2 y = u$, $a > 0$, where u is the input and y is the output, both scalars. **9 marks**

(a) Obtain a state-space model using the states $x_1 = y$, $x_2 = \dot{y}$.

(b) Investigate the stability of the system.

(c) Show that the model obtained in (a) is controllable.

(d) Design a control law $u = -Kx$ to place the eigenvalues at $-a, -a$. ← specification

(e) Show that the model obtained in (a) is observable.

(f) Design an observer to estimate the state from output measurements. Describe the overall controller.

(g) Suppose instead of state estimation and state feedback, direct output feedback, $u = -ky$ is used. Is the specification given in (d) achievable? Explain.

2. According to the Popov-Belevitch-Hautus test,

$\text{rank}\{[B \; AB \ldots A^{n-1}B]\} = n \Leftrightarrow \text{rank}\{[\lambda I - A \;\; B]\} = n$ $\forall$ complex numbers λ.

A is $n \times n$ matrix

B is $n \times m$ matrix

Verify this for

(a) $A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

8 marks

(b) $A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} -1 & -2 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

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(3) Suppose $\dot{x} = Ax + Bu$ and an invertible co-ordinate transformation $z = Tx$ is made to obtain $\dot{z} = \tilde{A}z + \tilde{B}u$.

(a) Find $\tilde{A}$ and $\tilde{B}$. $\boxed{\text{5 marks}}$

(b) Show that $\text{rank}\{[\tilde{B} \ \tilde{A}\tilde{B} \ \ldots \tilde{A}^{n-1}\tilde{B}]\} = \text{rank}\{[B \ AB \ldots A^{n-1}B]\}$.

(c) Suppose $\tilde{A} = \begin{array}{c} \left[\begin{array}{c|c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hline 0 & \tilde{A}_{22} \end{array}\right] \begin{array}{l} \updownarrow r \\ \updownarrow n-r \end{array} \\ \overset{\longleftrightarrow}{r} \quad \overset{\longleftrightarrow}{n-r} \end{array}$, $\tilde{B} = \left[\begin{array}{c} \tilde{B}_1 \\ \hline 0 \end{array}\right] \begin{array}{l} \updownarrow r \\ \updownarrow n-r \end{array}$ and

$\text{rank}\{[\tilde{B}_1 \ \tilde{A}_{11}\tilde{B}_1 \ \ldots \tilde{A}_{11}^{r-1}\tilde{B}_1]\} = r$. What is $\text{rank}\{[B \ AB \ldots A^{n-1}B]\}$?

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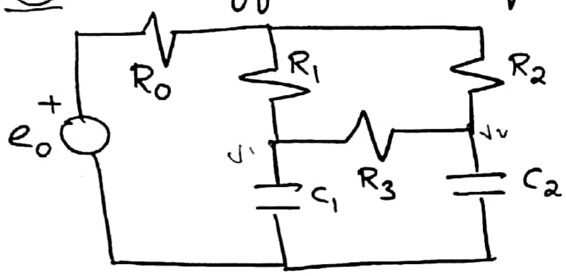
(4) The differential equations for the bridge circuit



are $\dot{v}_1 = -\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_3}\right)v_1 + \frac{1}{C_1 R_3}v_2 + \frac{1}{C_1 R_1}e_0$,

$\dot{v}_2 = \frac{1}{C_2 R_3}v_1 - \frac{1}{C_2}\left(\frac{1}{R_2} + \frac{1}{R_3}\right)v_2 + \frac{1}{C_2 R_2}e_0$.

If the bridge is balanced $R_1 C_1 = R_2 C_2$, show that the state-space model with state $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and input e_0 is uncontrollable. $\boxed{\text{5 marks}}$

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(5) "Ideas of controllability and observability were introduced by R.E. Kalman in the mid 1950's as a way of explaining why a method of designing compensators for unstable systems by cancelling poles in the right half-plane by zeros in the right half-plane is doomed to fail even if the cancellation is perfect. Kalman showed that a perfect pole-zero cancellation would result in an unstable system with a stable transfer function. The transfer function, however, is of lower order than

the system, and the unstable modes are either not capable of being affected by the input (uncontrollable) or not visible in the output (unobservable)."

Consider the system below with input u and output y,

$$\dot{x}_1 = 2x_1 + 3x_2 + 2x_3 + x_4 + u,$$

$$\dot{x}_2 = -2x_1 - 3x_2 - 2u, \qquad \boxed{8 \text{ marks}}$$

$$\dot{x}_3 = -2x_1 - 2x_2 - 4x_3 + 2u,$$

$$\dot{x}_4 = -2x_1 - 2x_2 - 2x_3 - 5x_4 - u,$$

$$y = 7x_1 + 6x_2 + 4x_3 + 2x_4.$$

The transfer function from $u \to y$ is $H(s) = \frac{1}{s+1}$.

(a) Obtain a state-space model with variables $x = [x_1 \;\; x_2 \;\; x_3 \;\; x_4]^T$.

(b) Transform the co-ordinates into $z = Tx$,

$$T = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix},$$

and obtain the state-space model in terms of z.

(c) Using the transformed model, show that the system is neither controllable nor observable.

(d) Explicitly write down the differential equations of the transformed model $z = [z_1 \;\; z_2 \;\; z_3 \;\; z_4]^T$ and identify the variables that are affected by the input and the variables that are visible in the output.

—— X —— X —— X ——