ELL 705 MAJOR TEST

Duration: 2 hours | Total Marks: 30

Instructions
1. Show all relevant steps clearly and briefly.
2. If needed, make suitable assumptions. But, state them clearly.
3. For the general state-space model,

$$
\begin{aligned}
x_{k+1} &= F_k x_k + G_k w_k, \\
z_k &= H_k' x_k + v_k,
\end{aligned}
$$

where v_k and w_k are assumed to be independent zero-mean white-noise processes with $E[v_k v_k'] = R_k \delta_{kl}$ and $E[w_k w_k'] = Q_k \delta_{kl}$, respectively, the Kalman Filter takes the form,

$$
\begin{aligned}
\hat{x}_{k+1|k} &= (F_k - K_k H_k')\hat{x}_{k|k-1} + K_k z_k, \\
K_k &= F_k \Sigma_{k|k-1} H_k [H_k' \Sigma_{k|k-1} H_k + R_k]^{-1}, \\
\Sigma_{k+1|k} &= F_k [\Sigma_{k|k-1} - \Sigma_{k|k-1} H_k (H_k' \Sigma_{k|k-1} H_k + R_k)^{-1} H_k' \Sigma_{k|k-1}] F_k' + G_k Q_k G_k', \\
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + \Sigma_{k|k-1} H_k [H_k' \Sigma_{k|k-1} H_k + R_k]^{-1} (z_k - H_k' \hat{x}_{k|k-1}), \\
\Sigma_{k|k} &= \Sigma_{k|k-1} - \Sigma_{k|k-1} H_k [H_k' \Sigma_{k|k-1} H_k + R_k]^{-1} H_k' \Sigma_{k|k-1}.
\end{aligned}
$$

Here, $x_k, z_k, v_k, w_k, F_k, H_k, G_k, Q_k, R_k, \Sigma_{k|k}$, and $\Sigma_{k+1|k}$ are standard matrices of appropriate dimension.

Question 1. *(9 marks)* Consider the process $x_{k+1} = a x_k + w_k, y_k = 0.5 x_k + v_k$, where x_k is the state, y_k is the output, a is an unknown (scalar) parameter and v_k, w_k are independent zero-mean Gaussian white-noise processes with $E[v_k v_k'] = R\delta_{kl}$ and $E[w_k w_k'] = Q\delta_{kl}$. Design an Extended Kalman Filter to estimate parameter a from output measurements.

Question 2. *(7 marks)* Consider the process $x_{k+1} = 0.5 x_k + w_k, y_k = 0.5 x_k + v_k$, where x_k is the state, y_k is the output, and v_k, w_k are independent zero-mean Gaussian white-noise processes with $E[v_k v_k'] = R\delta_{kl}$ and $E[w_k w_k'] = Q\delta_{kl}$. It is desired to design a Kalman Filter to estimate the state from output measurements.
i) Show that the covariance propagation equation for $\Sigma_{k|k-1}$ satisfies the equation $P_{k+1} = P_k R/(P_k + 4R) + Q$, where $P_k = \Sigma_{k|k-1}$.
ii) Assuming that P_k converges to a constant value P, find P?
iii) Does P_k converge to P?

Question 3. *(6 marks)* Consider the points (0,1), (1,2), and (3,3) on the X-Y plane. Note that they do not lie on a straight line. Find the equation of the straight line that minimizes the sum of the distances to each of these points. 7/5 8/5

$V(m,c) = (1 - m\times 0 - c)^2 + (2 - m\times 1 - c)^2 + (3 - m\times 3 - c)^2$ $y = mx + c$ m, c as para

Find m, n that minimizes V

Question 4. *(8 marks)* Make the following geometric interpretation to stochastic variables: For stochastic variables, x_s and y_s, the vectors are represented by x_g and y_g, respectively. The covariance between x_s and y_s is the dot (scalar) product between x_g and y_g, $cov(x_s, y_s) = x_g.y_g$. The length of x_g is given by $||x_g|| = \sqrt{x_g.x_g} = \sqrt{cov(x_g, x_g)}$. The angle θ between the vectors x_g and y_g is given by,

$$\cos\theta = \frac{x_g.y_g}{||x_g||||y_g||} = \frac{cov(x_s, y_s)}{\sqrt{cov(x_s, x_s)cov(y_s, y_s)}}.$$

Therefore, if x_s and y_s are uncorrelated, or independent with zero mean, then x_g and y_g are orthogonal vectors. With this interpretation in mind, we will drop the subscripts s and g, taking the stochastic or geometric interpretation depending on the context.

Consider the process $x_{k+1} = 0.5x_k + w_k, y_k = 0.5x_k + v_k$, where x_k is the state, y_k is the output, and v_k, w_k are independent zero-mean Gaussian white-noise processes with $E[v_k v_k'] = R\delta_{kl}$ and $E[w_k w_k'] = Q\delta_{kl}$.

The error in the predicted value of y_k is,

$$\begin{aligned}\tilde{y}_k &= y_k - \hat{y}_{k|k-1},\\ &= 0.5x_k + v_k - 0.5x_{k|k-1},\\ &= 0.5\tilde{x}_{k|k-1} + v_k.\end{aligned}$$

Here, $\tilde{x}_{k|k-1}$ does not depend on v_k and so they are uncorrelated, implying that the corresponding vectors are orthogonal.

i) Sketch the triangle formed by the vectors corresponding to $\tilde{y}_k$, $0.5\tilde{x}_{k|k-1}$ and v_k.

The state update can be written as,

$$\begin{aligned}\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_k\tilde{y}_k,\\ \Rightarrow \tilde{x}_{k|k} &= \tilde{x}_{k|k-1} - K_k\tilde{y}_k,\end{aligned}$$

where $\tilde{x}_{k|k} = x_k - \hat{x}_{k|k}$, $\tilde{x}_{k|k-1} = x_k - \hat{x}_{k|k-1}$, and K_k is the Kalman gain to be determined.

ii) Sketch the triangle formed by the vectors corresponding to $\tilde{x}_{k|k}$, $K_k\tilde{y}_k$, and $\tilde{x}_{k|k-1}$.

iii) From the two triangle sketched above and other geometric considerations, show that $\tilde{x}_{k|k}$ is minimum when K_k is $||\tilde{x}_{k|k-1}||^2/||\tilde{y}_k||^2$.

iv) Using the Pythagoras Theorem on the triangle formed by $\tilde{x}_{k|k}$, $K_k\tilde{y}_k$, and $\tilde{x}_{k|k-1}$, find a relation between $P_{k|k}$ and $P_{k|k-1}$, where $P_{k|k} = cov(\tilde{x}_{k|k}, \tilde{x}_{k|k})$ and $P_{k|k-1} = cov(\tilde{x}_{k|k-1}, \tilde{x}_{k|k-1})$.