August 27, 2016

Maximum Marks: 25

1. Let $P : (x_1, y_1)$ and $Q : (x_2, y_2)$ be two points in $\mathbb{R}^2$. Let the metric d be defined as

$$d(P,Q) = Max(|x_1 - x_2|, |y_1 - y_2|).$$

(i) Show that $(\mathbb{R}^2, d)$ is a metric space.

(ii) Let $P_0 : (x_0, y_0) \in \mathbb{R}^2$. Sketch the ball of radius δ with centre P_0 in $(\mathbb{R}^2, d)$.

(iii) Is $\left\{\left(\frac{1}{n}, \frac{2}{n}\right), n \in \mathbb{N}\right\}$ a sequence in $(\mathbb{R}^2, d)$? Is it convergent? Find its limit. [5]

2. Let V be the set of all (2×2) real matrices and $M \subseteq V$ be

$$M = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \; b = c \text{ and } a + d = 0 \right\}.$$

Here $a, b, c, d \in \mathbb{R}$.

(i) Show that V is a real vector space under the usual operations of addition and scalar multiplication of matrices.

(ii) Verify that M is a vector subspace of V.

(iii) Identify a basis of M.

(iv) Write the dimension of M. [5]

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y + z \\ x + y - z \end{pmatrix}.$$

(i) Verify that T is a linear transformation.

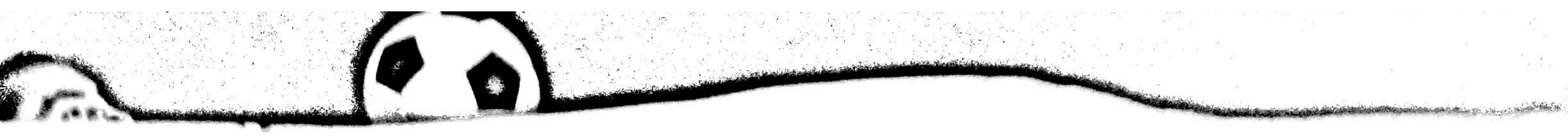
(ii) Let a basis for $\mathbb{R}^3$ be given as

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Also let a basis for $\mathbb{R}^2$ be given as

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Determine the matrix A of the linear transformation T. [5]



4. Let $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$. Show that

$$(|x_1| + ... + |x_n|)^2 \leq n(|x_1|^2 + ... + |x_n|^2).$$

(Hint: Use the Cauchy-Schwartz inequality.) (5)

5. Give an example of each of the following (if no such example is possible, give reasons for the same).
(i) A (3×3) real symmetric matrix having eigenvalues as 1, $(1 + i)$, and $(1 - i)$.
(ii) A Cauchy sequence in $(\mathbb{R}^3, d)$ which is not convergent. Here d is the Euclidean distance in $\mathbb{R}^3$.
(iii) A cone in $\mathbb{R}^3$ which is not convex.
(iv) A countable set M in $\mathbb{R}$ such that $\overline{M} = \mathbb{R}$. (Here $\overline{M}$: Closure of M). (5)
(v) A (3×3) matrix A such that $Rank\ A = 3$ and A^{20} is singular.