ALGEBRA - MAL516

MAJOR TEST

Maximum Credit : 40 May 2, 2015

The numbers on the right indicate maximum credit for the corresponding problems. JUSTIFY YOUR ANSWERS.

Q.1. Let R be a commutative ring with 1. Show that the following conditions are equivalent.

(i) R has a unique maximal ideal.

(ii) All non-units of R are contained in some ideal $M \neq R$.

(iii) All non-units of R form an ideal.

(iv) For all $r, s \in R$, $r+s=1$ implies either r is a unit or s is a unit. [10]

Q.2. Let R be a ring such that for each sequence of ideals $A_1, A_2, \ldots$ of R with $A_1 \subseteq A_2 \subseteq \ldots$, there exists a positive integer n (depending on the sequence) such that $A_m = A_n$ $\forall\, m > n$. Let $f : R \to R$ be a ring epimorphism. Show that $\mathrm{Ker} f = \{0\}$. [5]

Q.3. Let F be a finite field of characteristic 2 and a be a non-zero element of F. Is the polynomial $x^2 - a$ irreducible in $F[x]$? [4]

Q.4. Prove that no finite field is algebraically closed. [4]

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Q.5. Show that the following conditions on a field F are equivalent.

(a) Every non-constant polynomial $f \in F[x]$ has a root in F.

(b) Every non-constant polynomial $f \in F[x]$ splits over F.

(c) Every irreducible polynomial in $F[x]$ has degree 1.

(d) There is no algebraic extension field of F (except F itself).

(e) There exists a subfield K of F such that F is algebraic over K and every polynomial in $K[x]$ splits in $F[x]$. [12]

Q.6. Let F be an extension field of K, E an intermediate field and H a subgroup of $\mathrm{Aut}_K F$. Show that

(i) $H' = \{v \in F : \sigma(v) = v \ \forall \sigma \in H\}$ is an intermediate field of the extension. [3]

(ii) $E' = \{\sigma \in \mathrm{Aut}_K F : \sigma(u) = u \ \forall u \in E\}$ is a subgroup of $\mathrm{Aut}_K F$. [2]

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