MAL 741 (Fractal Geometry)
(Major Test 2015)

Max Time: 2 hours

Max Marks: 40

1. (a) Define s-dimensional Hausdorff outer measure $\mathcal{H}^s$. [2]

   (b) Show that for any E, $\sup\{s \geq 0 : \mathcal{H}^s(E) = \infty\} = \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\}$. [3]

2. (a) Define the Hausdorff metric on the set of all non-empty closed and bounded subsets of a metric space. [2]

   (b) Prove that the Hausdorff metric satisfies the triangle inequality. [2]

   (c) What is the distance between the intervals $[0,1]$ and $[2,3]$ of $\mathbb{R}$ in the Hausdorff metric. [1]

3. (a) Let $E = \cup_{k=1}^{\infty} L_k$, where L_k is the line segment $\{(x, \frac{1}{\sqrt{k}}) : 0 \leq x \leq \frac{1}{\sqrt{k}}\}$ in $\mathbb{R}^2$. Show that $\underline{\lim}_B E \geq 4/3$. [4]

   (**Hint**: You may need to obtain the inequality $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \geq 2\sqrt{n+1} - 2$.)

   (b) What is the Hausdorff dimension of E. [1]

4. Let $f : [0,1] \to \mathbb{R}$ be a function satisfying $|f(t) - f(u)| \leq c|t-u|^{2-s}$ for all $t, u \in [0,1]$, where $c > 0$ and $1 \leq s \leq 2$. Show that $\overline{\dim}_B$ graph $f \leq s$ and $\mathcal{H}^s(\text{graph } f) < \infty$. [5]

5. Let μ be a mass distribution on $\mathbb{R}^n$, let $F \subset \mathbb{R}^n$ be a Borel set and let $0 < c < \infty$ be a constant. Prove that if $\overline{\lim}_{r \to 0} \mu(B(x,r))/r^s < c$ for all $x \in F$, then $\mathcal{H}^s(F) \geq \mu(F)/c$. [4]

6. Let $f(z)$ be a polynomial of degree at least 2. Prove that $J(f)$ is non-empty and compact. [6]

7. Prove that $J(f)$ contains all repelling periodic points of f for any polynomial of degree ≥ 2. [4]

8. (a) Let $f(z) = z^2 + 2z$. Determine the Julia set $J(f)$. [2]

   (b) Let $f_c(z) = z^2 + c$ and $|c| \leq \frac{1}{4}$. Show that $\overline{B(0, \frac{1}{2})} \subset K(f_c) \subset B(0,2)$. [4]