MCL361 Minor Exam (16 Feb 2022)

Avoid skipping steps. Cheating shall be heavily penalized. Max. marks: 30

For Questions $1-5$, consider n random variables $X_1, X_2, \cdots, X_n$ where all the X_i 's are independent exponentially distributed random variables having identical parameter λ.

1. (0.5 mark) From elementary probability theory, we know that if X and Y are two nonnegative random variables having PDFs f_X and g_Y, then $Z = X + Y$ has PDF $h_Z(z) = \int_0^z f_X(z-b)g_Y(b)db$. Thus, show steps to compute the PDF of the newly defined random variable $S_2 = X_1 + X_2$.
2. (1 mark) Also show steps to compute the PDF of the newly defined random variable $S_3 = X_3 + S_2$.
3. (0.5 mark) Define the random variable $S_n = X_1 + X_2 + \cdots + X_n$. Then using previous answers, guess the PDF for S_n.
4. (3 marks) Prove your previous conjecture by induction.
5. (3 marks) Following is a well-known formula for indefinite integral.

$$\int z^n e^{\lambda z} dz = e^{\lambda z} \sum_{i=0}^{n} (-1)^{n-i} \frac{n!}{i!\lambda^{n-i+1}} z^i + C$$

   where C is the constant of integration. Use it to show that the CDF of Erlang distribution is

$$1 - \sum_{i=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^i}{i!}$$

6. (2 marks) In the $M/M/c$ queue, compute the probability that an arriving customer has to wait in queue in terms of λ, μ, c, p_0.
7. (3 marks) Customers are arriving at a facility according to a Poisson process with arrival rate λ. Each customer will be independently retained in the facility for further processing with probability p or rejected otherwise. We have seen during lectures that an arrival process is a Poisson process with rate λ if the probability of 0 arrival in a small interval h is $1 - \lambda h$, probability of 1 arrival is λh, and probability of 2 or more arrivals is negligible. Make use of this to show that both the arrivals of relevant as well as rejected customers can be modelled as Poisson processes and find their corresponding rate parameters.
8. (2 marks) Men are arriving at a facility according to a Poisson process with arrival rate λ_1. Women are arriving at a facility according to a Poisson process with arrival rate λ_2. Similar to previous question, show that the combined arrival of both men and women can be modelled as a Poisson process and find its corresponding rate parameter.
9. (2 marks) For the previous combined arrival process, assume that it is known that an arrival has happened in the facility. Then find the conditional probability of that arrival being that of a man.
10. (2 marks) Consider a facility modeled as an $M/M/1$ queue which is busy on average for five minutes out of every six. The mean service time is 45 seconds. If the time spent waiting for service to begin is equal to 3 minutes, compute the mean number of customers in the queueing system using Little's law.

For Questions $11-17$, consider the EPL model. Assume the production rate to be P units per unit time. Let the lot (order) size be denoted by y units. Assume the demand rate to be D units per unit time. Assume that $P > D$. Let K be the setup cost associated with the placement of an order given in INR per order. Let h be the holding cost in INR per inventory unit per unit time. The reorder point should be set to zero as in the classical EOQ model. Assume that as soon as the order is placed, production starts. Also assume that as soon as a unit is produced, it arrives at the inventory facility; there is no delivery lag.

11. (1 mark) Similar to EOQ model, plot the inventory level against time assuming there was no unit in the inventory facility to begin with. Indicate the slopes of all the lines shown in the plot.
12. (0.5 mark) What is the time required to produce a single lot?
13. (0.5 mark) What is the maximum inventory size that can occur in EPL model?

14. (1 mark) What is the time period of each inventory cycle?

15. (1 mark) What is the holding cost per cycle?

16. (1 mark) What is the total inventory cost per unit time $\text{TCU}(y)$?

17. (1 mark) What lot size minimizes $\text{TCU}(y)$? Do justify that the obtained stationary point is indeed a minimum point.

18. (3 marks) Recall the newsvendor model discussed during lectures. Consider the case with both fixed and variable setup costs. Now suppose that you are interested in maximizing the net expected profit instead of minimizing the total expected cost. To compute profit, subtract the setup costs and holding costs from the revenue generated. Assume that $r > 0$ is the revenue generated per unit item and $h > 0$ is the holding costs per unit item during the single period considered. Show that the objective function in this case is the same as that for the cost-based formulation of the newsvendor model up to a constant term.

19. (1 mark) What is the special connection between lead time and steady state probability in $(s-1, s)$ model with Poisson demands?

20. (1 mark) In the probabilistic EOQ model, we saw that y^* and R^* are related as follows

$$y^* = \sqrt{\frac{2D(K + pS)}{h}} \qquad\qquad \int_{R^*}^{\infty} f(x)dx = \frac{hy^*}{pD}$$

So if we increase y^*, does R^* increase or decrease? To justify your answer, compute the relevant derivative.