Department of Mathematics
MTL 100: Calculus
2014-15: Semester I
Major Exam
25 November 2014

Please begin the answer to each question on a new page, and give adequate explanation for full credit.

1. Let $p : \mathbb{N} \to \mathbb{N}$ be such that $p(m) < p(n)$ whenever $m < n$. Let $\sum a_n$ and $\sum b_n$ be two series of real numbers related as follows:

$$\begin{aligned} b_1 &= a_1 + a_2 + \cdots + a_{p(1)}, \\ b_{n+1} &= a_{p(n)+1} + a_{p(n)+2} + \cdots + a_{p(n+1)} \text{ for } n \geq 1. \end{aligned}$$

Assume that there is a constant $M > 0$ such that $p(n+1) - p(n) < M$ for all n, and assume that $\lim_{n\to\infty} a_n = 0$. Prove that $\sum a_n$ converges if and only if $\sum b_n$ converges, and that they converge to the same sum when they do. **[5]**

2. (a) Let f be a real-valued bounded function on $[a, b]$. For subsets S of $[a, b]$, let

$$\Omega_f(S) = \sup\left\{f(x) - f(y) : x \in S, y \in S\right\}.$$

For $x \in [a, b]$, define

$$\omega_f(x) = \lim_{\delta\to 0} \Omega_f\Big((x - \delta, x + \delta) \cap [a, b]\Big).$$

Prove that if f is continuous at x_0, then $\omega_f(x_0) = 0$. **[2]**

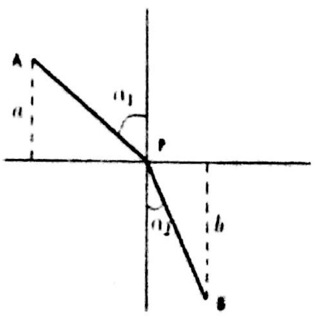
(b) Let A be a non-empty subset of $\mathbb{R}$. Define $d_A : \mathbb{R} \to \mathbb{R}$ by

$$d_A(x) = \inf\left\{|x - a| : a \in A\right\}$$

for each $x \in \mathbb{R}$. Prove that d_A is uniformly continuous on $\mathbb{R}$. **[3]**

3. The Fermat Principle in optics states that the path APB taken by a ray of light in passing across the plane separating two different optical media is such that the travel time t is minimized. If v_1 is the velocity when lights travels along AP and v_2 the velocity when lights travels along PB, use this principle to deduce the Law of Refraction:

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{v_1}{v_2}.$$



[5]