Department of Mathematics
MTL 100: Calculus
2016-17: Semester I
Major Exam

21 November 2016

Please begin the answer to each question on a new page, and give adequate explanation for full credit.

1. Two sequences of positive integers $\{a_n\}$ and $\{b_n\}$ are defined recursively by taking $a_1 = b_1 = 1$ and equating rational and irrational parts in the equation

$$a_n + b_n\sqrt{2} = (a_{n-1} + b_{n-1}\sqrt{2})^2 \quad \text{for } n \geq 2.$$

Prove that $a_n^2 - 2b_n^2 = 1$ for $n \geq 2$. Deduce that $\{a_n/b_n\} \downarrow \sqrt{2}$ and that $\{2b_n/a_n\} \uparrow \sqrt{2}$. **[2+3]**

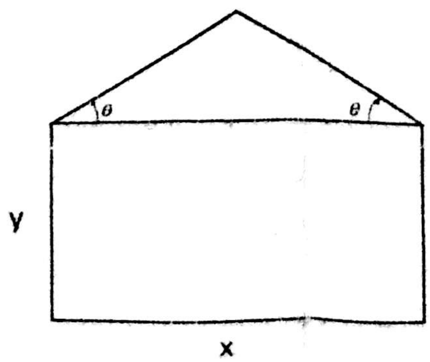
2. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $|f(x) - f(y)| \leq \alpha|x - y|$ for all $x, y \in \mathbb{R}$ and some constant $\alpha \in (0,1)$. Prove that there is a *unique* $x_0 \in \mathbb{R}$ for which $f(x_0) = x_0$. **[6]**

3. Determine the interval and the radius of convergence for the power series

$$\frac{1}{2} + \frac{1}{3}z + \frac{1}{2^2}z^2 + \frac{1}{3^2}z^3 + \frac{1}{2^3}z^4 + \frac{1}{3^3}z^5 + \cdots.$$

[4]

4. A pentagon is composed of an isosceles triangle on top of a rectangle (see figure below). Given that the perimeter of the pentagon is P, use tests concerning maxima and minima involving functions in x, y, θ to find its dimensions that *maximize* the area. **Do not use the method of Lagrange multipliers.** **[5]**



5. Define a function $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0; \\ \frac{1}{2^n} & \text{if } \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n}, n \in \{0,1,2,3,\ldots\}. \end{cases}$$

For each positive integer n, define a partition $\mathscr{P}_n$ of $[0,1]$ where $\Delta x_i = \frac{1}{2^n}$ for each i.

(a) Given $\epsilon > 0$, prove that there exists a positive integer N such that

$$\mathcal{U}(f; \mathscr{P}_n) - \mathcal{L}(f; \mathscr{P}_n) < \epsilon$$

whenever $n > N$.

(b) Let $F : [0,1] \to \mathbb{R}$ be given by $F(x) := \int_0^x f(t)\, dt$. Show that for $0 < x \leq 1$ we have

$$F(x) = x\, I(x) - \tfrac{1}{3}\, I(x)^2,$$

where $I(x) = 2^{\lceil \log_2 x \rceil}$ and $\lceil x \rceil$ denotes the least integer greater than or equal to x. **[2+3]**

6. Consider the integral $\displaystyle\int_0^\infty \frac{x|\log x|^p}{x^2+1}\, dx$, $p \in \mathbb{R}$. Giving reason(s) for the integral being improper, determine whether the integral converges or diverges. **[1+4]**

7. Evaluate

$$\iint_{\mathcal{R}} \sin(x-y)\cos(x+2y)\, dx\, dy,$$

where $\mathcal{R}$ is the parallelogram bounded by the lines $x - y = 0$, $x - y = \pi$, $x + 2y = 0$, and $x + 2y = \pi/2$. **[4]**

8. Verify Green's theorem for

$$\int_{\partial\mathcal{R}} (x+y^2)\, dx + x^2 y\, dy,$$

where $\partial\mathcal{R}$ is the boundary of the region $\mathcal{R}$, containing the point $(\frac{1}{4}, 0)$, and is bounded by the curves $y^2 = x$ and $|y| = 2x - 1$. **[6]**