MTL 100: Calculus
Major: 2017-18 Semester I

Department of Mathematics

36

Total marks: 45
Max Time: 2 hrs

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1. Using the linear transformation $u = x - y$ and $v = x + y$, evaluate the double integral

$$\iint_R (x-y)^2 \sin^2(x+y)\,dx\,dy,$$

where R is the parallelogram with vertices $(\pi, 0), (2\pi, \pi), (\pi, 2\pi), (0, \pi)$. [5]

2. Find the maximum of the function $f(x,y,z) = \log x + \log y + 3\log z$ on that portion of the sphere $x^2 + y^2 + z^2 = 20$ where $x > 0, y > 0, z > 0$. [4]

3. (a) Test the convergence of the improper integral

$$\int_0^\infty (\log x)^2 e^{-x^2}\,dx.$$

(b) Show that $\displaystyle\int_0^1 (x - x\sqrt{x})^{\frac{1}{2}}\,dx = \frac{32}{105}$. [4+3]

4. (a) Let f be a continuous function on $[a,b]$ and define a function $F : [a,b] \to \mathbb{R}$ by

$$F(x) := \int_a^x f(t)\,dt.$$

Then prove that F is differentiable on (a,b) and for every $x \in (a,b)$, $F'(x) = f(x)$.

(b) Prove that if g is Riemann integrable on $[a,b]$, then $|g|$ is also Riemann integrable on $[a,b]$. [4+4]

5. Find the linear approximation of the function $f(x,y) = x^2 + xy + y^2$ at the point $(1,1)$ and also estimate the approximation error in the region
$R = \{(x,y) : |x-1| \le 1, |y-1| \le 1\}$. [5]

6. Consider the function $f(x,y) = \begin{cases} \dfrac{x^2 y|y|^\alpha}{x^4 + y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

Show that

(a) f is continuous at $(0,0)$ for all $\alpha > 0$;

(b) f is differentiable at $(0,0)$ for all $\alpha > 1$. [3+3]

7. Test the uniform continuity of the function $f(x) = x\sin x$ on $[0, \infty)$. [4]

8. Which of the following statements are TRUE/FALSE: Justify your answer.

(a) If a real sequence $\{x_n\}$ satisfies $|x_{n+1} - x_n| < \dfrac{1}{n}$ for every $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

(b) The sequence $x_n = \left(1 - \dfrac{1}{n}\right)\cos\dfrac{n\pi}{2}, n \in \mathbb{N}$, is convergent. [3+3]