

Indian Institute of Technology Delhi  
Department of Mathematics  
MTL 100 (Calculus)- Major exam: Semester I 2021 – 22

Date: March 2, 2022

Total Marks: 40

Duration: 2 hours

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MARKS WILL BE AWARDED ONLY FOR THOSE ANSWERS WITH PROPER JUSTIFICATION

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**Question 1:** Let  $R$  be the region in the first quadrant bounded by the hyperbolas  $xy = 1$ ,  $xy = 9$  and the lines  $y = x$  and  $y = 4x$ . Using the transformations  $u = \sqrt{xy}$  and  $v = \sqrt{\frac{y}{x}}$ , evaluate  $\iint_R \left( \sqrt{\frac{y}{x}} + \sqrt{xy} \right) dA$ . [4]

**Question 2:**

(a) Determine the radius of convergence of the power series  $\sum_{n=1}^{\infty} \frac{2n^2 + n - 1}{3n + 4} x^{2n}$ .

(b) Discuss the convergence of the series  $\sum_{n=1}^{\infty} \frac{n \sin(n\pi/2)}{n^2 + 1}$ . [3+3]

**Question 3:** Assume that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a four times differentiable function on  $\mathbb{R}$  which satisfies  $xf(x) = \int_0^x \frac{dt}{1+t^4}$  for all  $x$  in  $\mathbb{R}$ . Find the fourth order Taylor polynomial of  $f$  about  $x = 0$ . [4]

**Question 4:** For  $p, q > 0$ , discuss the convergence of the improper integral [5]

$$\int_0^{\infty} \frac{e^{-x}}{x^p(1-x)^q} dx.$$

**Question 5:** Let  $\alpha > 0$  and  $f(x, y) = \begin{cases} \frac{\sin(x^2)|y|^\alpha}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$  [5]

Find all values of  $\alpha$  such that  $f$  is differentiable at  $(0, 0)$ . Justify your answer.

**Question 6:** Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a continuous function. Show that there exists a point  $\alpha \in [0, 1]$  such that [4]

$$\int_0^1 x^2 f(x) dx = \frac{1}{3} f(\alpha).$$

**Question 7:** Let  $(a_n)_{n \in \mathbb{N}}$  be a bounded sequence. Show that for  $p > 0$  the sequence  $(s_n)_{n \in \mathbb{N}}$  converges to zero, where [4]

$$s_n = \frac{a_n}{n^{p+1}} \left( \frac{\log(1)}{1} + \frac{\log(2)}{2^p} + \dots + \frac{\log(n)}{n^p} \right).$$

**Question 8:** Suppose  $f : [0, 1] \rightarrow [0, \infty)$  is a three times differentiable function on  $(0, 1)$ . If  $f(x) = 0$  for at least two values of  $x$  in  $(0, 1)$ , prove that  $f'''(c) = 0$  for some  $c$  in  $(0, 1)$ . [4]

**Question 9:** Discuss the uniform continuity of  $f(x) = \sin(\sqrt{x}) \log(x)$ ,  $x \in (0, \infty)$ . [4]