INDIAN INSTITUTE OF TECHNOLOGY DELHI
DEPARTMENT OF MATHEMATICS

MAL 111 MAJOR TEST

MAXIMUM CREDIT: 40 DATE: 29/11/2006 (WEDNESDAY)

JUSTIFY ALL YOUR ANSWERS.

1. Let $f : [0,1] \to \mathbb{R}$ be a Riemann integrable function satisfying the condition
$$|f(x) - f(y)| \leq 4|x-y| \quad \text{for all } x, y \in [0,1].$$
Show that $|\int_0^1 f(t)dt - f(c)| \leq 2$ for all $c \in [0,1]$.

[5]

2. Let $f : \mathbb{R} \to [0, \infty)$ be a decreasing function. For $n \in \mathbb{N}$, define
$$a_n = f(1) + f(2) + \cdots + f(n) - \int_1^n f(t)dt.$$
   (a) Show that $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$.
   (b) Show that $a_n \geq 0$ for all $n \in \mathbb{N}$.
   (c) Does the sequence $(a_n)_{n=1}^{\infty}$ converge in $\mathbb{R}$? Justify your answer.

[3+3+2]

3. Let U be an open subset of $\mathbb{R}^2$ and $f : U \to \mathbb{R}$ be a function defined on U. Suppose that $f_x(x,y)$ and $f_y(x,y)$ exist at each point (x,y) in U. Moreover, assume that $f_x(x,y) = 0 = f_y(x,y)$ for all $(x,y) \in U$. Is f constant on U? Justify your answer.

[5]

4. (a) Let $f : (X,d) \to (Y,\rho)$ be a continuous function between two metric spaces (X,d) and (Y,ρ), and let A be a nonempty connected subset of X. Show that $f(A)$ is connected in (Y,ρ).
   (b) Let (X,d) be a metric space and A and B be two nonempty connected subsets of X such that $A \cap B \neq \emptyset$. Show that $A \cup B$ is also connected in (X,d).

[5+5]

5. Let (X,d) be a metric space and $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be two Cauchy sequences in (X,d). Show that the sequence $\left(d(x_n, y_n)\right)_{n=1}^{\infty}$ converges in $\mathbb{R}$. ($\mathbb{R}$ has the usual distance metric.)

[5]

6. Let $f(x) = e^x$ for $x \in \mathbb{R}$. Find the Taylor series for f about 0. For which values of $x \in \mathbb{R}$ does this Taylor series converge to $f(x)$? Justify your answer.

[4]

7. Find the maximum and minimum values of the function $f(x,y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

[5]