INDIAN INSTITUTE OF TECHNOLOGY DELHI

MAL 111: Introduction to Analysis and Differential Equations

MAJOR TEST 2008-09 (I Semester)

Maximum Marks: 50 Time: 2 Hours

Give complete statements of the results used.

Justify all your answers.

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(1) (a) Let X be the set of positive integers with metric

$$d(m,n) = \left|\frac{1}{m} - \frac{1}{n}\right| \quad \text{for all } m, n \in \mathbb{N}.$$

Show that (X, d) is not a complete metric space.

(b) Let (X, d_X) and (Y, d_Y) be two metric spaces and let A be a non-empty subset of X. Let $f : X \to Y$ and $g : X \to Y$ be continuous functions such that $f(x) = g(x)$ for every $x \in A$. Show that $f(x) = g(x)$ for every $x \in \overline{A}$.

(c) Show that every compact metric space has the Bolzano-Weierstrass property.

[2+3+5]

(2) (a) Evaluate the limit $\lim_{x \to \infty} (1 + \frac{1}{x^2})^x$.

(b) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. For $x \in [a, b]$, define

$$F(x) = \int_a^x f(t)\, dt.$$

Show that F is differentiable on (a, b) and $F'(x) = f(x)$ for all $x \in (a, b)$.

(c) Find the length of one arch of the cycloid

$$x = a(t - \sin t), y = a(1 - \cos t).$$

[3+4+3]

(3) (a) Let

$$f(x,y) = \begin{cases} 0 & \text{if } xy \neq 0, \\ 1 & \text{if } xy = 0. \end{cases}$$

Show that $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exist but f is not continuous at $(0,0)$.

(b) Show that $(\frac{\pi}{3}, \frac{\pi}{3})$ is a critical point of the function

$$f(x,y) = \sin x \sin y \sin(x+y).$$

Investigate this function for extreme values at this point.

(c) Find the extreme values of the function $f(x,y,z) = xy + z^2$ on the circle in which the plane $y - x = 0$ intersects the sphere $x^2 + y^2 + z^2 = 4$. **[2+4+4]**

(4) (a) Show that the function $f(x,y) = |\sin y| + x$ satisfies a Lipschitz condition in the y-variable on the whole xy-plane but $\frac{\partial f}{\partial y}$ does not exist when $y = 0$.

(b) Apply Picard's iteration to find approximate solutions to the problem

$$y' = x + y, \quad y(0) = -1.$$

Do three steps of iteration.

(c) Find the general solution of the diffrential equation $y'' - 5y' + 6y = x^4$.

(d) Solve the diffrential equation $y^{(5)} - y^{(4)} + 2y''' - 2y'' + y' - y = 0$.

(e) Find the general solution of the equation $x^3y''' - 3x^2y'' + 6xy' - 6y = 24x^5$.

[4+3+5+3+5]