

Max Time: 2 hours

Max Marks: 40

Note: No marks will be awarded without appropriate arguments.

1. Let $u_1 = (1, 1, -1)$, $u_2 = (1, -1, 1)$, $u_3 = (2, 3, 4)$, $u_4 = (a, 1, 6)$ and $v_1 = (1, 2, 3)$, $v_2 = (2, 3, 4)$, $v_3 = (3, 4, 5)$, $v_4 = (10, 15, b)$. Find $a, b \in \mathbb{R}$ such that there is a linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ satisfying $T(u_i) = v_i$ for $i = 1, 2, 3, 4$. [4]

[6 = 2 × 3]

2. Justify whether the following statements are true or false.

(a) Let $W_1 = \{(x, 2x, 3x) \in \mathbb{R}^3 : x \in \mathbb{R}\}$ and $W_2 = \{(2x, 3x + y, 3x - y) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$. Then $\mathbb{R}^3 = W_1 \oplus W_2$.

(b) There exists a linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $\text{range}(T) = \text{ker}(T)$.

(c) The matrix $\begin{pmatrix} x & x \\ y & 2y \end{pmatrix}$ is diagonalizable over \mathbb{R} for all $x, y \in \mathbb{R}$.

3. Consider the IVP: $y' = x + y$, $y(0) = 0$.

[5 = 2 + 3]

(a) Find the solution of the given IVP.

(b) Using the Picard's approximation, find a formula for the n-th iterate $y_n(x)$ for the above IVP. Also, show that $y_n(x)$ converges to the solution obtained in (a).

4. Find the general solution of the following ODE using the power series method.

[5]

$$y'' - 2xy' - 2y = 0.$$

5. Find the inverse Laplace transform of

[4]

$$F(s) = \frac{1}{(s^2 + 4)(s^2 + 2s + 2)}$$

6. Show, using the properties of Laplace transform, that

$$s^2 + 2s + 2 = (s^2 + 4) + 2s - 2$$

[4]

$$\mathcal{L} \left[\frac{e^t}{n!} \frac{d^n}{dt^n} (t^n e^{-t}) \right] = \frac{(s-1)^n}{s^{n+1}}$$

7. Solve the following IVP

[6]

$$y'' + 4y' + 5y = [1 - u(t - 10)]e^t - e^{10}\delta(t - 10); \quad y(0) = 0, y'(0) = 1.$$

8. Let $Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, $A = \begin{pmatrix} -4 & 12 \\ -3 & 8 \end{pmatrix}$ and $g(t) = e^{2t} \begin{pmatrix} 6t \\ 1-t \end{pmatrix}$.

[6 = 3 + 3]

(a) Find two linearly independent solutions for the homogeneous system $Y' = AY$.

(b) Use variation of parameters formula, to find a particular solution for the nonhomogeneous system $Y' = AY + g(t)$.

$$y_1 = \int_0^x f(t) y_2 dt \quad \dots \text{END} \dots$$

$y_1 = e^{-x} + 2e^{-x}$
 $y_2 = e^{-x} + xe^{-x}$
 $\frac{dy}{dx} = x + y$
 $dy - y = x$
 $\int \frac{dy}{y} = \int \frac{x}{y} dx$