MTL101: Major Exam, Semester I, 2016-17

November 21, 2016

Max Marks: 40 **Max Time: 2 Hrs**

You can do questions in any order but must do both parts of a question together at one place.

1. (a) Let $V = \mathbb{R}_+$ be a set of all positive real numbers. For $x, y \in V$, $\lambda \in \mathbb{R}$, define the following three operators:

$$x \oplus y = xy, \qquad \lambda \odot x = x^\lambda, \qquad \lambda \otimes x = \lambda^x.$$

State weather the following statements are TRUE or FALSE with correct justification.

(i) The operator $\odot$ distributes over $\oplus$.

(ii) The operator $\otimes$ distributes over $\oplus$.

(iii) Closure property holds in V with respect to $\otimes$.

(iv) There exists $v \in V$ such that $V = \text{span}\{v\}$ with respect to $\odot$.

(b) Let $V = M_{3\times 3}(\mathbb{R})$, the vector space of 3×3 real matrices over the field of reals. Let

$$W = \Big\{A = [a_{ij}] \in V \mid \sum_{i=1}^{3} a_{ij} = 0,\ j = 1, 2, 3\Big\}.$$

Prove that W is a subspace of V. Construct a basis of W. [4+4]

2. (a) In vector space $\mathbb{R}^8$ over the field $\mathbb{R}$, prove that the intersection of any three subspaces, each subspace of dimension 6, can not be the zero subspace.

(b) Let A be an $n \times n$ real matrix and I be the identity matrix of order n.

(i) Prove that $N(A) \subseteq Range(I - A)$.

(ii) Is $Range(I - A) \subseteq N(A)$?

(iii) Suppose $A^2 = A$. Is $Range(A) \bigcap Range(I - A) = \{0\}$?

Here, $N(M)$ and $Range(M)$ denote the null space and the range space of a matrix M, respectively. [4+4]

3. (a) Let V and W be two finite dimensional vector spaces over the same field, and $T : V \to W$ be a linear transformation. Prove that

$$\dim(V) = \text{rank}(T) + \text{nullity}(T).$$

(b) Does there exist a linear transformation T from $\mathbb{R}^5$ to $\mathbb{R}^2$ whose null space equals $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1 = 3x_2,\ x_3 = x_4 = x_5\}$. Give reason. [6+2]

$u_2 = v_2 - v_1$
$2v_1 - v_2$

4. (a) Let V be a 2-dimensional vector space and $T : V \to V$ be a linear transformation. If $B = \{v_1, v_2\}$ and $B_1 = \{u_1, u_2\}$ are any given ordered bases in V such that $v_1 = u_1 + u_2$ and $v_2 = u_1 + 2u_2$ and $[T]_{B_1} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$, then find $[T]_B$.

$Tu_1 = 2u_1 + 3u_2$
$Tu_2 = u_1 + 4u_2$

$\boxed{Tv_1 = 3u_1 + 7u_2 \quad Tv_2 = 4u_1 + 11u_2}$

(b) Let V be the space of all real sequences over the field of reals. Define two ~~linear trans~~formations $T_1, T_2 : V \to V$ as

$$T_1(x_1, x_2, x_3, x_4 \ldots) = (2x_2, 3x_3, 4x_4, \ldots), \quad T_2(x_1, x_2, x_3, x_4 \ldots) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4} \ldots).$$

Is T_1 one-one? Is T_2 onto? Are the two compositions $T_1 \circ T_2$ and $T_2 \circ T_1$ bijections? Justify your answers. [4+4]

5. (a) Let $A = \begin{pmatrix} 1 & 3 & -1 \\ 9 & 2 & 1 \\ 7 & 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 5 \\ 7 \\ 13 \end{pmatrix}$. Show that the system $Ax = b$ is inconsistent. Find the set of all least square solutions to $Ax = b$.

(b) Let x, y and z be positive real numbers. Use Cauchy-Schwarz inequality to prove that

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{x+y+z}{2}.$$

No marks will be awarded if the result is proved without using Cauchy Schwarz inequality. [5+3]