MTL 101
LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS

MAJOR: PART A

Notation: $x'(t), x''(t)$ and $x'''(t)$ denote the first, second and third derivatives of function x with respect to variable t, respectively.

PART A: 10:00 to 11:20.
No Submission is allowed after 11:20.
If you submit PART A with PART B, it will not be graded.

Question 1: (3 Marks) Find the orthogonal trajectories of the family of curves $x = ce^{t^2}$.

Question 2: (5 Marks) Consider the IVP

$$\frac{dx}{dt} = \frac{\cos x}{1 - t^2}, \quad x(0) = x_0,$$

with $x_0 \in \mathbb{R}$. Find the largest interval $|t| < h$ on which the existence and uniqueness theorem guarantees a unique solution.

Question 3: (4 Marks) Solve the following ODE

$$(xdt + tdx)\, t \cos\left(\frac{x}{t}\right) = (tdx - xdt)\, x \sin\left(\frac{x}{t}\right).$$

Question 4: (6 Marks) Consider the differential operator with constant coefficients

$$L = \frac{d^2}{dt^2} - 2\frac{d}{dt} + 1$$

and let $f(t) = \sin(t) + e^t + t$.

(a) Determine M, a linear differential operator with constant coefficients, such that it annihilates f.

(b) Taking into account **part(a)**, find the general solution of $L(x(t)) = f(t)$ using the method of undetermined coefficients.

Question 5: (8 Marks) Consider the differential equation

$$t^3 x''' + t^2 x'' - 2x = 1 \tag{1}$$

(a) Reduce (1) into an ODE with constant coefficients and find the corresponding first order system.

(b) Find the fundamental matrix of the first order system obtained in **part(a)**.

(c) Using the method of variation of parameters for the first order system obtained in **part(a)**, find the general solution of (1).

END OF PART A

MTL 101
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MAJOR: PART B

Notation: $x'(t), x''(t)$ and $x'''(t)$ denote the first, second and third derivatives of function x with respect to variable t, respectively.

PART B: 11:45 to 13:05.
No Submission is allowed after 13:05.

DO NOT UPLOAD PART A WITH PART B. IT WILL NOT BE GRADED.

Question 1: (6 Marks) Discuss the existence and uniqueness of the solutions for the following IVPs:

(a)

$$\begin{aligned} x_1' &= |x_1|^p + t^2, \quad p \geq 1 \\ x_2' &= \tan^{-1}(x_2), \\ x_1(1) = 0&, \quad x_2(1) = 0. \end{aligned}$$

(b)

$$\begin{aligned} x_1' &= |x_2|^p + t^2, \quad 0 < p < 1 \\ x_2' &= \sin(x_1), \\ x_1(1) = 0&, \quad x_2(1) = 0. \end{aligned}$$

Question 2: (5 Marks) Solve the following IVP using Laplace Transform:

$$x'' + 3x' + 2x = \sin(t) + \delta(t-2)$$

with initial conditions $x(0) = 0$ and $x'(0) = 5$. Here $\delta(t-2)$ is the Dirac Delta function centred at $t = 2$. Check if the solution is continuously differentiable (C^1).

Question 3: (6 Marks) Find the inverse Laplace Transform of the following functions:

$$\text{(a) } \ln\left(1 - \frac{a^2}{s^2}\right), \qquad \text{(b) } \frac{s^2}{(s^2+a^2)^2}.$$

Question 4: (7 Marks) Consider the following linear system of equations:

$$\begin{aligned} x_1' &= -2x_1 + x_2 + 4x_3 \\ x_2' &= -5x_1 + 2x_2 + 5x_3 \\ x_3' &= -x_1 + x_2 + \lambda x_3 \end{aligned}$$

If $\vec{x} = \vec{a}te^{2t}$ is a solution of this system for some constant vector $\vec{a}$, then find the general solution of the system.

END OF PART B