MTL 101
LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS
MAJOR EXAM

Total: 40 Marks **Time: 2:00 Hrs.**

Question 1: (5 Marks) Let $u, v \in \mathbb{R}^n$ be two fixed vectors such that $\langle u|v\rangle \neq 0$, where $\langle \cdot|\cdot\rangle$ is the standard inner product on $\mathbb{R}^n$. Define $T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(x) = \langle x|v\rangle u$. Then

(a) Show that T is a linear operator on $\mathbb{R}^n$.

(b) Prove/disprove: T is diagonalizable.

Question 2: (5 Marks) Let V be a vector space over $\mathbb{C}$ with $\dim(V) = 2$. Suppose $T : V \to V$ has eigenvalues 1 and 2. For all positive integer n, show that

$$T^n = 2^n(T - I) - (T - 2I).$$

Here I is the identity operator on V.

Question 3: (5 Marks) Consider the vector space $V = M_{n\times n}(\mathbb{C})$ of all $n\times n$ complex matrices over $\mathbb{C}$ and let $0 \neq A \in V$. Prove that the set $\{I, A, \dots, A^n\}$ is not a part of any basis of V.

Question 4: (5 Marks)

(a) Discuss the Existence and Uniqueness of the solutions of the following Initial Value Problem:

$$x'(t) = t^2 + \sqrt{\tan\big(t + x(t)\big)}, \quad x\left(\frac{\pi}{6}\right) = 0.$$

(b) Let $x(t)$ be a nonnegative function with

$$x(t) \leq L \int_{t_0}^{t} x(s)\ \mathrm{d}s,$$

where $t_0 \leq t$ and L is a nonnegative constant. Is it true that $x(t) = 0$ for all t? Justify.

Question 5: (5 Marks) Let $f_1, f_2 \in C^1(I)$ (set of all continuously differentiable functions on open interval I) be given functions such that $W(f_1, f_2)(t) = 0$ for all $t \in I$. Show that there exists an interval $I_0 \subseteq I$ such that f_1 and f_2 are linearly dependent on I_0.

Question 6: (5 Marks) For $\alpha \in \mathbb{R}$, let

$$L(x(t)) = t^3x''' + 3(1-\alpha)t^2x'' + (3\alpha^2 - 3\alpha + 1)tx' - \alpha^3x, \quad t > 0.$$

(a) Reduce the differential equation $L(x(t)) = 0$ into a differential equation with constant coefficients, say $M(x(t)) = 0$.

(b) Find the general solution of differential equation $M(x(t)) = 0$.

(c) For $\alpha = 1$, use variation of parameters method to find the general solution of $L(x(t)) = 1$.

Question 7: (5 Marks) Find the general solution of the system of ODEs:

$$X' = AX$$

with

$$A = \begin{pmatrix} 8 & 12 & -2 \\ -3 & -4 & 1 \\ -1 & -2 & 2 \end{pmatrix}.$$

Question 8: (5 Marks) Find the general solution of

$$x'' + tx' + x = 0$$

using series solution methods.