MTL101: Minor I, Semester I, 2016-17
August 30, 2016

Max Marks: 20 **Max Time: 1 hr**

Attempt all question. You can choose to answer the questions in any order. Strictly do all parts of the same question together at one place in your answer script.

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1. Let $V = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$, and the field $F = \mathbb{R}$. Define two operations on V as follows:

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 - y_2)$$

$$\alpha(x_1, x_2) = \begin{cases} (0,0) & \text{if } \alpha = 0 \\ \left(\alpha x_1, \dfrac{x_2}{\alpha}\right) & \text{if } \alpha \neq 0 \end{cases} \qquad \alpha \in F.$$

State whether each of the following statements is **TRUE** or **FALSE**. Give reason to support your answers. No marks will be awarded if the answer of the statement is not justified.

(a) Vector addition (defined by $+$) in V is associative but not commutative.

(b) $(0, 0)$ is the additive identity in V.

(c) $\alpha\,(v + w) = \alpha\, v + \alpha\, w, \quad \forall\, \alpha \in F, \quad \forall\, v, w \in V.$

(d) $(\alpha + \beta)\, v = \alpha\, v + \beta\, v, \quad \forall\, \alpha, \beta \in F, \quad \forall\, v \in V.$ [4]

2. Prove that the linear span of a non-empty finite subset S of a vector space V over the field F is the smallest subspace of V containing S. [4]

3. Let U, S, W be any three subspaces of a vector space. Prove or disprove (with counter-example) the following:

(a) $U \bigcap (S + W) \subseteq (U \bigcap S) + (U \bigcap W)$.

(b) $(U \bigcap S) + W \subseteq (U + W) \bigcap (S + W)$ [4]

4. Let $V = M_{2\times 2}(\mathbb{C})$ be the vector space of all 2×2 matrices with complex entries over the field $F = \mathbb{R}$. Let W be a subset of V consisting of all symmetric matrices whose sum of the principal diagonal elements is zero, that is,

$$W = \{X \in V \mid X^t = X, \ \text{trace}(X) = 0\}.$$

(a) Prove that W is a subspace of V.

(b) Show that $V = W \bigoplus \widetilde{W}$, where $\widetilde{W}$ is the subspace of V consisting of all matrices whose first row has entries equal to zero. [4]

5. Which of the following sets of vectors are linearly independent/dependent over the specified field?

(a) $\{(1, 1, -1, -1), (0, 1, 1, 0), (2, 0, 2, -2)\} \subset (\mathbb{Z}_3)^4$, $F = \mathbb{Z}_3$, the field of integer modulo 3.

(b) $\{1 + i, 1 - i, 2 + \sqrt{3}, 2 - \sqrt{3}\} \subset \mathbb{C}$, $F = \mathbb{Q}$, the field of rational numbers. [4]