MTL 101
LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS
MINOR EXAM

Total: 20 Marks **Time: 1:00 Hrs.**

Question 1 (**4 Marks**)

a) If A and B are two $n \times n$ real matrices such that $AB = 5I_{n\times n}$, then is it true that $BA = 5I_{n\times n}$? If true, give a proof else, give a counterexample.

b) Let A be a 3×4 real matrix of rank 3. Show that there exists 4×3 real matrix B such that $AB = I_{3\times 3}$.

Question 2 (**3 Marks**) Consider the vector space $P_3(\mathbb{R})$ of polynomials of degree less than or equal to three with real coefficients.

a) Prove that $\mathcal{B} = \{1 - x, 1 + x^2, 1 - x^3, 1 + x - x^3\}$ is a basis for $P_3(\mathbb{R})$.

b) Find the coordinates of the vector $u = 1 + x + x^2 + x^3$ with respect to ordered basis $\mathcal{B}$.

Question 3 (**4 Marks**) Consider the vector space $\mathbb{C}^2(\mathbb{C})$. Find all possible linear transformations $T : \mathbb{C}^2 \to \mathbb{C}^2$ such that $T^2 := T \circ T : \mathbb{C}^2 \to \mathbb{C}^2$ (the composition of T with itself) is given by

$$T^2(z_1, z_2) = (-z_1 + 2z_2, -z_2) \text{ for all } (z_1, z_2) \in \mathbb{C}^2.$$

Question 4 (**4 Marks**)

a) Let W_1, W_2 be non-zero subspaces of a finite dimensional vector space V over $\mathbb{C}$. Suppose that there exists $f : V \to \mathbb{R}$ such that $f(w_1) - f(w_2) < 0$ for all non-zero vectors $w_1 \in W_1$ and $w_2 \in W_2$. Prove that $\dim W_1 + \dim W_2 \leq \dim V$.

b) Let W_1, W_2 be subspaces of the vector space $\mathbb{R}^4$ over $\mathbb{R}$ given by

$$\begin{aligned} W_1 &= \operatorname{span}\{(4,3,2,1), (1,1,1,2), (3,2,1,-1)\} \\ W_2 &= \operatorname{span}\{(1,0,3,2), (4,3,2,1)\} \end{aligned}$$

Find the dimension of $W_1 + W_2$.

Question 5: (**5 Marks**) Consider the linear operator $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$T((x_1, x_2, x_3, x_4)) = \Big(\sum_{i=1}^{4} x_i, \sum_{i=1}^{4} x_i, \sum_{i=1}^{4} x_i, \sum_{i=1}^{4} x_i\Big) \quad \text{for all} \quad (x_1, x_2, x_3, x_4) \in \mathbb{R}^4.$$

Prove or disprove: there exists an ordered basis B of $\mathbb{R}^4$ such that $[T]_B$ is diagonal.