MTL 101
LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS
RE-MINOR

Total Marks: 20 **Time: 70 Minutes**

Question 1: Let $\mathcal{P}_5(\mathbb{R})$ denote the space of all polynomials of degree less than or equal to 5 with real-coefficients. Consider the linear transformation $L : \mathcal{P}_5(\mathbb{R}) \to \mathbb{R}^6$ defined by

$$L(f) = \big(f(1), f(2), f(3), f(4), f(5), f(6)\big).$$

a) (1 Mark) Find the kernel of L.
b) (1 Mark) Is L bijective (one-one and onto) ?
c) (3 Marks) Using your answers to items (a), (b) above, answer the following questions (Do not use any other method).
Does there exist a polynomial $f(x) \in \mathcal{P}_5(\mathbb{R})$ such that

$$f(1) = 1,\ f(2) = 9,\ f(3) = 2,\ f(4) = 0,\ f(5) = 2 \text{ and } f(6) = 0?$$

If so, how many?

Question 2: (5 Marks) Let V be a finite dimensional inner product space over $\mathbb{R}$ and $T : V \to V$ be a map such that
a) $T(0) = 0$
b) $\|T(x) - T(y)\| = \|x - y\|$ for all $x, y \in V$, where $\|.\|$ denotes the norm (length) induced by the inner product.

Prove that T is a linear transformation and $\langle T(x)|T(y)\rangle = \langle x|y\rangle$ for all $x, y \in X$.
Does the conclusion hold if the condition $T(0) = 0$ is dropped ? Justify.

Question 3: Let $\mathcal{P}(\mathbb{C})$ be the vector space of all polynomials with complex coefficients over $\mathbb{C}$. Let T be a function from $\mathcal{P}(\mathbb{C})$ into $\mathcal{P}(\mathbb{C})$ such that

$$T(p(x) + q(x)) = T(p(x)) + T(q(x)), \quad T(\alpha p(x)) = \overline{\alpha}\, T(p(x)), \quad \text{and} \quad T^2(p(x)) = p(x),$$

for all $p(x), q(x) \in \mathcal{P}(\mathbb{C})$ and $\alpha \in \mathbb{C}$. Then,
a) (2 Marks) Show that $U = \{p(x) \in \mathcal{P}(\mathbb{C}) :\ T(p(x)) = p(x)\}$ is a vector space over $\mathbb{R}$ with respect to operations defined on $\mathcal{P}(\mathbb{C})$.
b) (3 Marks) Prove that $\mathcal{P}(\mathbb{C}) = U \oplus iU$, where $i = \sqrt{-1}$.

Question 4: Let $f : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be defined by $f(x, y) = \sum_{j=1}^{n} x_j \overline{y}_{n+1-j}$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then
a) (2 Marks) Prove or disprove: f defines an inner product on $\mathbb{C}^n$.
b) (1+2 Marks) Let $M = \operatorname{span}(\{e_1\})$, where $e_1 = (1, 0, \ldots, 0) \in \mathbb{C}^n$, and consider the subspace $M^o = \{v \in \mathbb{C}^n :\ f(v, w) = 0 \text{ for all } w \in M\}$. Prove or disprove that $\mathbb{C}^n = M \oplus M^o$. Find a basis for M^o. Justify.