MTL101
LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS
MINOR 2

Total Marks: 30 **Time: 90 Minutes**

Question 1:

a) (4 Marks) Consider the vector space $V(\mathbb{R})$ spanned by functions $\{e^{5x}, \sin(x), \cos(x), e^{3x}\}$ with usual sum of functions and scalar multiplication. Let $T : V \to V$,

$$T(f) = \frac{d^2 f}{dx^2} - 8\frac{df}{dx} + 15f.$$

Find the basis for null space (Kernel) and range space of T.

b) (3 Marks) Consider the vector space $V(\mathbb{R})$ of all polynomials of degree less than or equal to 3. We define $T : V \to V$ as

$$T(p(x)) = \frac{d^2 p(x)}{dx^2} + \frac{dp(x)}{dx} + p(x).$$

Find the matrix representation of T i.e. $[T]_B$ with respect to the basis $B = \{1, 2x, x + x^2, x^3\}$.

Question 2: Prove or disprove the following statements.

a) (2 Marks) There exists a matrix A such that $\{(1, 7, 8)\}$ is a basis for the column space of A and $\{(8, 7, 1)\}$ is a basis for the null space of A.

b) (2 Marks) There exists a matrix A such that $\text{Nullity}(A) = 1 + \text{Nullity}(A^T)$.

Question 3: Let $V(\mathbb{R})$ be the vector space of all polynomials of degree less than or equal to 2. Consider the matrix

$$P = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

a) (2 Marks) Find a basis B for V so that the matrix P given above is the change of basis matrix corresponding to the basis change from B to $B' = \{2x^2+3x+1, 2x^2+2x+1, -x^2-2\}$.

b) (2 Marks) If $B = \{2x^2 + 3x + 1, 2x^2 + 2x + 1, -x^2 - 2\}$, then find a basis B' for V so that P is the change of basis matrix from B to B'.

Question 4: (3 marks) Let T: $\mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator whose matrix representation with respect to the standard basis is

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ -2 & -3 & 0 \end{bmatrix}.$$

Find the eigenvalues and eigenvectors of T. What can you say about diagonalizability of T?

Question 5: (4 marks) Let $V(\mathbb{R})$ be a finite dimensional vector space and $T : V \to V$ be a linear operator on V such that $T(v)$ and v are linearly dependent for every $v \in V$. Find the explicit form of T. Prove or disprove that T is diagonalizable.

Question 6:

a)(4 Marks) Define an inner product on $\mathbb{R}^2$ such that

$$\langle e_1 | e_2 \rangle = -1 \quad \text{and} \quad \langle e_1 | e_1 \rangle = 2,$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Justify your answer.

b) (4 Marks) Consider $\mathbb{C}^3$ with the standard inner product. Find an orthonormal basis for the subspace spanned by the vectors $v_1 = (1, 0, i)$ and $v_2 = (2, 1, 1 + i)$.