MTL-102 Differential Equations
Department of Mathematics, IIT Delhi
Major Exam, (May 2016)

Time: 2 Hours Max. Marks: 50

1. Find a simple wave solution $u(x,y) = v(x/y)$ for $(G(u))_x + u_y = 0$ when $G(u) = u^4/4$. Use this to define a continuous weak solution of $(G(u))_x + u_y = 0$ for $y > 0$ that satisfies

$$u(x,0) = \begin{cases} 0 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \end{cases}$$

[7]

2. Consider $u = u_x^2 + u_y^2$ with the initial condition $u(x,0) = ax^2$. For what positive constant a there exists a solution ? Is it unique ? Find all solutions. [5]

3. Derive fundamental solution of one dimensional heat equation [7]

$$u_t = u_{xx}, |x| < \infty, t > 0.$$

4. Consider the ODE $y''' + y'' + y' + y = 0$,
(a) What system of the first order ODEs is equivalent to this equation ?
(b) If the system in (a) is denoted as $\mathbf{y}' = \mathbf{f}(x, \mathbf{y})$, find a Lipschitz constant K for this f to satisfy a Lipschitz condition on the set $S : |x| < \infty, |\mathbf{y}| < \infty$.
(c) Let ϕ be any solution of the ODE in the above. Then $\Phi = (\phi, \phi', \phi'')$ is a solution of the system of first order equations. Show that if x_0 is any real number then [8]

$$|\Phi(x)| \le |\Phi(x_0)| e^{K|x-x_0|}.$$

5. Prove that every initial value problem for the below system

$$y_1' = y_1 + x^{10} y_2 + y_3,$$
$$y_2' = e^{2x} y_1 + (\cos x) y_3,$$
$$y_3' = 10 y_1 - e^{-10x} y_2 - 5 y_3,$$

has a unique solution which exists for all real x. [5]

6. Write the statement of Sturm Comparison theorem and use it to discuss the zeros of a nontrivial solution of the Bessel's equation $x^2 y'' + xy' + (x^2 - 4)y = 0$ on the positive $x-$axis. [7]

7. Discuss the nature and stability properties of the critical point $(0,0)$ for each of the following linear autonomous systems:

(a) $\frac{dx}{dt} = 4x - 2y$, $\frac{dy}{dt} = 5x + 2y$. (b) $\frac{dx}{dt} = 5x + 2y$, $\frac{dy}{dt} = -17x - 5y$. [6]

8. Write the statement of Liapunov's theorem and use it to prove that $(0,0)$ is an asymptotically stable critical point of the nonlinear system $\frac{dx}{dt} = -y - x^3$, $\frac{dy}{dt} = x - y^3$. [5]