Department of Mathematics
MTL102: Differential Equations
II Semester 2017-18
Max Marks: 50

Major | Max Time: 2 Hours

Answer ALL questions in both pages 1 and 2.

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1. Classify the following first order PDEs into linear, semilinear, quasilinear or fully nonlinear PDEs: [3 Marks]

(a) $u_x + u_y + u_z = u$ (b) $u_x^2 + u_y^2 + u_z^2 = 1$ (c) $xu_x + yu_y + zu_z = u$

(d) $(u^2)_x + (u^2)_y + (u^2)_z = zu$ (e) $u_x + u_y + u_z = \sqrt{u}$ (f) $2u_x + 3u_y + 4u_z = 5.$

2. Find a simple wave solution $u(x,t) = v(\frac{x}{t}), t > 0$ for the following PDE [5 Marks]

$$u_t + [F(u)]_x = 0$$

when $F(u) = \frac{u^2}{2}$. Use this to define a continuous weak solution of the above PDE for $t > 0$ that satisfies the initial condition

$$u(x,0) = \begin{cases} 0, & \text{for} \quad x > 0 \\ -2, & \text{for} \quad x < 0. \end{cases}$$

3. (a) Use Greens' Identity to prove [5 Marks]

$$\int_{\Omega} K(|x|)\Delta^2 u \quad dx = \int_{\partial\Omega} \left[K(|x|)\frac{\partial \Delta u}{\partial \nu} - u\frac{\partial \Delta K(|x|)}{\partial \nu} + \Delta K(|x|)\frac{\partial u}{\partial \nu} - \Delta u\frac{\partial K(|x|)}{\partial \nu}\right] ds,$$

where $K(|x|)$ is biharmonic i.e. $\Delta^2 K(|x|) = 0$ inside Ω.

(b) Solve the following PDE to derive all radial solutions $K(|x|)$:

$$(\Delta + a^2)K(|x|) = 0 \quad \text{in} \quad \mathbb{R}^3 - \{0\}$$

where $a > 0$. [3 Marks]

4. Find the general solution of the linear system: [6 Marks]

$$\frac{dx}{dt} = 16y, \frac{dy}{dt} = -x.$$

Find the critical points, the differential equation of paths and the equation of the paths. Sketch few of the paths with directions of increasing t and determine the nature and the stability of the critical points.

[Please Turn Over for Page-2.]

5. Write the first order systems equivalent to the following equations and discuss the stability properties of the critical point $(0,0)$: [3+3 Marks]

(a) The van der Pol equation with μ any real number:

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0.$$

(b) The pendulum equation with c, m, g, a are all positive real numbers:

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{g}{a}\sin x = 0.$$

6. Let $\phi(x)$ be a nontrivial solution of [5 Marks]

$$y'' + q(x)y = 0,$$

where $q(x) < 0 \quad \forall \quad x$. Then prove that $\phi(x)$ vanishes at most at one point.

7. Let $\phi_a(x)$ be a nontrivial solution of Bessel's equation [7 Marks]

$$x^2y'' + xy' + (x^2 - a^2)y = 0, x > 0.$$

Then prove the following:
(a) If $0 \leq a < \frac{1}{2}$ then every interval of length π contains at least one zero of $\phi_a(x)$.
(b) If $a = \frac{1}{2}$, then the distance between successive zeros of $\phi_a(x)$ is exactly π.
(c) If $a > \frac{1}{2}$ then every interval of length π contains at most one zero of $\phi_a(x)$.

8. (a) Give the definition of Lipschitz condition for a vector-valued function [3 Marks]

$$\underline{f}(x, \underline{y}) = (f_1(x, \underline{y}), f_2(x, \underline{y}), \cdots, f_n(x, \underline{y})) \quad \text{on} \quad S \subset \mathbb{R} \times \mathbb{C}^n.$$

(b) Given the vector-valued function defined on a rectangle $R = |x| \leq 2, |\underline{y}| \leq 1, (\underline{y} \in \mathbb{C}^3)$

$$\underline{f}(x, \underline{y}) = (xy_1 - (\sin x)y_2 + e^x y_3, (\cos x)y_1 - x^{10}y_2 + y_3, y_1 + y_2 + y_3).$$

Find the upper bound M for $|\underline{f}(x, \underline{y})|$ on the rectangle R. [2 Marks]
Does $\underline{f}(x, \underline{y})$ satisfy the Lipschitz condition? Justify your answer. [2 Marks]

(c) Given the coupled system of ODEs: [3 Marks]

$$y_1' = 1 + y_2^2$$
$$y_2' = x + y_1^2$$

with an initial condition $y_1(0) = 0$ and $y_2(0) = 0$. Compute the first three successive approximations to the solution of the system.