Department of Mathematics

MTL 103 (Optimization Methods and Applications)

Minor Exam 2

Time: 1 hour 10 minutes **Date: 18/08/20**
Max. Marks: 30

Note: The exam is closed-book, and all the questions are compulsory.

1. Consider a linear programming problem

$$\begin{aligned}&\min\ -3x_1 + x_2 + 3x_3 - x_4\\ &\text{s.t. } x_1 + 2x_2 - x_3 + x_4 = 0;\quad 2x_1 - 2x_2 + 3x_3 + 3x_4 = 9;\quad x_1 - x_2 + 2x_3 - x_4 = 6\\ &x_i \geq 0,\quad i = 1,2,3,4.\end{aligned}$$

   Let $x = (x_1, x_2, x_3, x_4)$ be a basic feasible solution where x_2, x_3, x_4 are basic variables and B is the associated basis. Is basis B an optimal basis? If B is not an optimal basis, find a cost reducing basic direction d and calculate the maximum improvement in the cost function by moving from x in the direction d. Find a new basic feasible solution by moving from x in the direction d. Is the new basic feasible solution an optimal solution of the linear program?

   (8 marks)

2. Let A be an $m \times n$ matrix, c be an $n \times 1$ vector, and $\mathbf{0}$ be an $m \times 1$ zero vector. Using duality theory prove that $Ax \leq \mathbf{0}$ implies $c^T x \leq 0$ if and only if $c^T = \lambda^T A$ for some $\lambda \geq \mathbf{0}$.

   (8 marks)

3. Consider a linear programming problem which we call it as a primal problem

$$\begin{aligned}&\min\ x_1 - 3x_2 - 2x_3\\ &\text{s.t. } 3x_1 - x_2 + 2x_3 \leq 7;\quad 2x_1 - 4x_2 \geq 12;\quad -4x_1 + 3x_2 + 8x_3 = 10\\ &x_1 \geq 0,\ x_2 \geq 0,\ x_3 \text{ is unrestricted}\end{aligned}$$

   a) Write the Lagrangian dual function for the primal problem and show that it gives a lower bound on the optimal cost of primal problem.
   b) Show that the best lower bound of the optimal cost of primal problem is given by the optimal value of another linear programming problem.

   (8 marks)

4. For each one of the following statements, state whether it is true or false. If true, provide a proof, else, provide a counterexample with proper explanation.

   - If a linear programming problem has infinite optimal solutions, then there exists at least two extreme points which are optimal solutions of the linear programming problem.
   - Suppose a linear programming problem always has an optimal solution. Then, there exists an extreme point which is an optimal solution.
   - Suppose x is a basic feasible solution of a linear programming problem in standard form such that the reduced cost associated with all non-basic variables is positive, i.e., $\bar{c}_j > 0$ for all $j \in N$, where N is the index set of non-basic variables. Then, x is a unique optimal solution.
   - Suppose a linear programming problem in standard form has infinite solutions and x is an optimal basic feasible solution such that the reduced cost for some non-basic variable is zero, i.e., $\bar{c}_j = 0$ for some $j \in N$. Then, there exists an optimal basic feasible solution different from x.

   (6 marks)