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INDIAN INSTITUTE OF TECHNOLOGY DELHI
DEPARTMENT OF MATHEMATICS
MTL104 (LINEAR ALGEBRA AND APPLICATIONS)
Minor I

Time: 1 hour **Maximum Marks: 35**

1. (a) (3 points) Suppose $u, v \in V$, a vector space over a field $\mathbb{F}$. If $\mathbb{F} = \mathbb{R}$, then show that $\{u, v\}$ is linearly independent if and only if $\{u + 2v, u - 2v\}$ is linearly independent.

   (b) (2 points) If $\{u, v, w\}$ is a linearly independent subset of a vector space, show that $\{u, u + v, u + v + w\}$ is also linearly independent.

2. (a) (1 point) Suppose V is a finite dimensional vector space and $T \in \mathcal{L}(V)$, give an example to show that V is not a direct sum of null space and range space of T. That is $V \neq \text{null}(T) \oplus \text{range}(T)$.

   (b) (4 points) Suppose $V = \mathbb{R}^n$ and $T \in \mathcal{L}(V)$ such that $Tv = Av$ where A is a $n \times n$ matrix with entries in $\mathbb{R}$. If $A^2 = A$, show that $V = \text{null}(T) \oplus \text{range}(T)$.

3. (5 points) Suppose $V = \mathbb{R}^3$ and $f_1, f_2, f_3 \in V' = \mathcal{L}(V, \mathbb{R})$ such that

$$f_i\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \sum_{j=1}^{3} (i + j)x_j \text{ for } 1 \leq i \leq 3.$$

   Let U be a subspace of V such that the annihilator of U, $U^\circ = \text{span}\{f_1, f_2, f_3\}$. Find a basis for U and its dimension.

4. (5 points) Suppose that V is a finite dimensional vector space over $\mathbb{C}$ and $T \in V' = \mathcal{L}(V, \mathbb{C})$. Let $v \notin \text{null}(T)$. Show that $V = \text{null}(T) \oplus \text{span}\{v\}$.

5. (5 points) Suppose that U is a subspace of V and the quotient space V/U is finite dimensional. Show that there exists a subspace W of V such that $\dim W = \dim V/U$ and $V = U \oplus W$.

6. (5 points) Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is such that T is injective and $\deg Tp \leq \deg p$ for every nonzero polynomial $p \in \mathcal{P}(\mathbb{R})$.

   (a) Prove that T is surjective.

   (b) Prove that $\deg Tp = \deg p$ for every nonzero $p \in \mathcal{P}(\mathbb{R})$.

7. (5 points) Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

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