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INDIAN INSTITUTE OF TECHNOLOGY DELHI
DEPARTMENT OF MATHEMATICS
MTL104 (LINEAR ALGEBRA AND APPLICATIONS)
Minor II

Time: 1 hour **Maximum Marks: 35**

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1. (a) (1 points) Suppose $S, T \in \mathcal{L}(V)$ and S is invertible. Suppose $p \in \mathcal{P}(\mathbb{F})$ is a polynomial. Prove that
$$p\left(STS^{-1}\right) = Sp(T)S^{-1}.$$
   (b) (4 points) Apply LU decomposition to solve the system $Ax = b$, where
$$A = \begin{pmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

2. (a) (2 points) Suppose $T, S \in \mathcal{L}(V)$ be linear operators on finite dimensional vector spaces V over $\mathbb{C}$. Suppose that $TS = ST$, then show that there is a eigenvector w that is common to both T and S.
   (b) (3 points) Let V be the vector space of $n \times n$ matrices with entries in $\mathbb{C}$. For a matrix $A \in V$ define a linear operator $T_A : V \to V$ such that $T_A(B) = AB$. If A is diagonalizable, show that T_A is diagonalizable.

3. (5 points) Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent:
   (a) $V = \operatorname{null} T \oplus \operatorname{range} T$.
   (b) $V = \operatorname{null} T + \operatorname{range} T$.
   (c) $\operatorname{null} T \cap \operatorname{range} T = \{0\}$.

4. (a) (3 points) Find an orthonormal basis for $\mathcal{P}_2(\mathbb{R})$ equipped with an inner product
$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx.$$
   (b) (2 points) What happens if the Gram–Schmidt Procedure is applied to a list of vectors that is not linearly independent?

5. (a) (2 points) Find the eigenvalues of the linear operator T on $\mathbb{R}^2$ which takes the circle $\{(x_1, x_2) | x_1^2 + x_2^2 = 1\}$ to the ellipse $\{(x_1, x_2) | x_1^2/a^2 + x_2^2/b^2 = 1\}$.
   (b) (3 points) Let T be the linear operator on $\mathcal{P}_2(\mathbb{R})$ defined by $T(f(x)) = f(1) + f'(0)x + (f'(0) + f''(0))x^2$. Check diagonalizability of this operator.

6. (a) (3 points) Suppose $\{v_1, \ldots, v_n\}$ form a linearly independent set of vectors. Show that there exists $w \in V$ such that $\langle w, v \rangle > 0$ for all $1 \le j \le m$.
   (b) (2 points) Show that the function that takes $((x_1, x_2), (y_1, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2$ to $|x_1y_1| + |x_2y_2|$ is not an inner product on $\mathbb{R}^2$.

7. (a) (3 points) Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U is invariant under T if and only if $U^\perp$ is invariant under the adjoint T^*.
   (b) (2 points) Show that $\dim \operatorname{null} T^* = \dim \operatorname{null} T + \dim W - \dim V$ for every $T \in \mathcal{L}(V, W)$.