Major exam

Course: MTL 105 **Duration: 2 hours** **M. Marks: 50**

Note: All questions are compulsory.

I. Let R be a commutative ring with unity. Show that a polynomial $f \in R[x]$ is a unit if and only if the constant term of f is a unit and all other coefficients of f are nilpotent. **[7 marks]**

II. Show that the additive group of $\mathbb{Z}[x]$ is isomorphic to the group of positive rational numbers under multiplication. **[6 marks]**

III. Show that 3 is an irreducible element of $\mathbb{Z}[\sqrt{-5}]$, but 3 is not prime in $\mathbb{Z}[\sqrt{-5}]$. Is $\mathbb{Z}[\sqrt{-5}]$ a unique factorization domain? **[3+2+2=7 marks]**

IV. If p is a prime number, then show that the polynomial $x^{p(p-1)} + x^{p(p-2)} + \cdots + x^p + 1$ is irreducible over $\mathbb{Q}$. **[5 marks]**

V. Let F be a finite field. Show that $F^* = F \setminus \{0\}$ is a cyclic group under multiplication. **[6 marks]**

VI. Show that the ring $\mathbb{Z}$ of integers is isomorphic to the ring of endomorphisms of the additive Abelian group of $\mathbb{Z}$. **[7 marks]**

VII. Let R be a ring. Show that every nilpotent ideal of R is nil, but the converse is not true. **[7 marks]**

VIII. Let $B \subseteq A$ be two ideals of a ring R. Show that $R/A \simeq \dfrac{R/B}{A/B}$. **[5 marks]**