DEPARTMENT OF MATHEMATICS
MTL 105: Algebra

Major *Marks - 50*

[Write each answer on a seperate page.]

1. **PROVE** or **DISPROVE** the following.
   [Marks are only for proper justification.]

   a) For a non-abelian group G, $a \in G$ is such that $o(a) = 7$. For any $b \in G$, if $a^4b = ba^4$ then $ab = ba$.

   b) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 8 & 9 & 6 & 1 & 5 & 3 & 2 & 10 & 7 & 4 \end{pmatrix} \in S_{10}$. Then $o(\sigma) = 10$.

   c) The center of $GL_n(\mathbb{R})$ is isomorphic to $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

   d) $GL_2(\mathbb{R})$ is a normal subgroup of $GL_2(\mathbb{C})$.

   e) Let $H = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\} < GL_2(\mathbb{C})$.
   Then H is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. [10 marks]

2. (a) Prove that the factor group $GL_3(\mathbb{Z}_7)/SL_3(\mathbb{Z}_7)$ is abelian.
   (b) Let G be a group with $o(G) = 6$. Prove that $Aut(G) \lhd G$ always and describe all non-isomorphic factor groups $G/Aut(G)$. [5+5 marks]

3. (a) Let ϕ and ψ be two homomorphisms from a group G to a group G'. Let $H = \{x \in G : \phi(x) = \psi(x)\}$. Prove that $H < G$.
   (b) Let G be an abelian group and $H < G$. Let $\phi : G \to H$ be a homomorphism such that $\phi(h) = h$, $\forall h \in H$.
   Prove that $G \cong H \times ker(\phi)$. [5+5 marks]

4. (a) Consider $H = \{\pm 1, \pm i\} < \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Prove that $\mathbb{C}^*/H \cong \mathbb{C}^*$.
   (b) For the ring $\mathbb{Z}$, determine all ring homomorphisms $\phi : \mathbb{Z} \to \mathbb{Z}$. [5+5 marks]

5. (a) Let D be an integral domain of characteristic 11. Define $\phi : D \to D$ as $\phi(a) = a^{11}$. Is ϕ a ring homomorphism? Justify.
   (b) Prove that the ring $\mathbb{Z}[i]/ < 1 + 3i >$ is isomorphic to $\mathbb{Z}_{10}$. [5+5 marks]