PROF- EKTA SAHA

Uploaded By- Rahul & Saurabh

2020MT10833

Rahul Kumar

MAJOR EXAMINATION (MTL105)
ACADEMIC SESSION: 2022-23
TOTAL MARKS: 50, DURATION: 2 HOURS

You need to mention all the statements/theorems that you use in your answers very clearly. If you use any statement/theorem in your answer that was not proved in the class/tutorial, you need to give a proof of that. Marks will be awarded on the basis of what you write in your answers, not on what you intend to write. Write your answers very neatly. If any of your answer is not readable, then it will not be graded.

---

(1) State the following statements are true/false with justification. **10×2=20**

(a) Sum of two nilpotent elements in a ring is always nilpotent.

(b) If $a \in R$ is a zero divisor then $a + I \in R/I$ is always a zero divisor where R is a ring and I a proper ideal of R.

(c) The ring $C([0,1], \mathbb{R})$ has uncountably many maximal ideals.

(d) In the ring $\mathbb{Z}[X]$, all the prime ideals are maximal ideals.

(e) There is no integral domain with infinitely many elements having non-zero characteristic.

(f) Quotient of a unique factorization domain is always a unique factorization domain.

(g) An element $a + \imath b \in \mathbb{Z}[\imath]$ is a unit if and only if $a^2 + b^2 = 1$.

(h) The polynomial ring $(\mathbb{Z}/2022\mathbb{Z})[X]$ is not a Euclidean domain.

(i) Let R be a Euclidean domain and $a, b \in R \setminus \{0\}$. Then $\gcd(a, b)$ exists.

(j) The polynomial $X^3 - 9$ is irreducible over $\mathbb{Z}/11\mathbb{Z}$.

(2) Prove that the field of fractions of $\mathbb{Z}[\sqrt{5}]$ is $\mathbb{Q}[\sqrt{5}]$. **3**

(3) Prove that $\mathbb{Q}[X]/\langle 1 + X^2 \rangle \simeq \mathbb{Q}[\imath]$. Use this to prove that $\mathbb{Q}[\imath]$ is a field. **3+2=5**

(4) Let R be a non-zero commutative ring with unity and I an ideal of R. Prove that $\mathrm{Rad}(I)$ is the intersection of all prime ideals of R containing I. **4**

(5) Let R be a UFD and K be its' field of fraction. Let $\alpha \in K$ be such that $f(\alpha) = 0$ where f is a monic polynomial over R. Prove that $\alpha \in R$. **4**

(6) Answer the following questions with complete justifications. **4+4+6=14**

(a) Is the polynomial $2X^4 + 52X^3 + 4044 \in \mathbb{Q}[X]$ irreducible?

(b) Does there exist a field with 121 elements?

(c) Does there exist an integral domain with 10 elements?

$\mathrm{Rad}(I) = \{a \in R : a^n \in I \text{ for } n \in \mathbb{N}\}$

Date: November 26, 2022.