Department of Mathematics
MTL 106/MAL 250 (Introduction to Probability and Stochastic Processes)
Major Test (II Semester 2014-15)

Time allowed: 2 hours **Max. Marks: 50**

1. The first generation of particles is the collection of off-springs of a given particle. The next generation is formed by the off-springs of these members. Assume particles act independently and identically irrespective of the generation. Suppose that the probability that a particle has k off springs (splits into k parts) is p_k, where $p_0 = 0.4$, $p_1 = 0.3$ and $p_2 = 0.3$.

   (a) Find the probability that there is no particle in the second generation. (2 marks)

   (b) Given that there are 500 particles in the 50th generation, what is the expected number of particles at 51th generation. (2 marks)

2. Let X be a random variable having the following cumulative distribution function:

$$F(x) = \begin{cases} 0, & x < -1 \\ \frac{1+x}{9}, & -1 \le x < 0 \\ \frac{2+x^2}{9}, & 0 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

   (a) Find $P(X \in E)$ where E is (i) $\{2\}$ (ii) $[-1/2, 3)$ (iii) $(-1, 0] \cup (1, 2)$. (3 marks)

   (b) Find the mean of X, if it exist? (2 marks)

3. Let X and Y be two independent continuous random variables.

   (a) Prove that $P[X \le Y] = \int_{-\infty}^{\infty} F_X(y) f_Y(y) dy$ where f_Y is the probability density function of Y and F_X is the cumulative distribution function of X. (2 marks)

   (b) Find the value of $P[X \le Y]$ when X and Y are i.i.d. random variables with common density function

$$f_X(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

   (2 marks)

4. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with mean 1 and variance 1600, and assume that these variables are non-negative. Let $Y = \sum_{k=1}^{100} X_k$. Use the central limit theorem to approximate the probability $P(Y \ge 900)$. Final answer can be in terms of $\Phi(z)$ where $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$. (4 marks)

5. In a communication system, the carrier signal at the receiver is modeled by $X(t) = \cos(2\pi w t + \theta)$ where θ is a uniform distributed random variable with interval $(-\pi, \pi)$ and w is a positive constant. Is $\{X(t),\ t \ge 0\}$ covariance/wide sense stationary? Justify your answer in details. (5 marks)

6. Two gamblers, A and B, bet on successive independent tosses of an unbiased coin that lands heads up with probability p. If the coin turns up heads, gambler A wins a rupee from gambler B, and if the coin turns up tails, gambler B wins a rupee from gambler A. Thus the total number of rupees among the two gamblers stays fixed, say N. The game stops as soon as either gambler is ruined; i.e., is left with no money! Assume the initial fortune of gambler A is i. Let X_n be the amount of money gambler A has after the nth toss. If $X_n = 0$, then gambler A is ruined and the game stops. If $X_n = N$, then gambler B is ruined and the game stops. Otherwise the game continues. Prove that $\{X_n, n = 0, 1, \ldots\}$ is a discrete time Markov chain. Write the one-step transition probability matrix or draw the state transition diagram for this Markov chain. (3 + 2 marks)

7. (a) Consider an aperiodic irreducible finite state space DTMC $\{X_n, n = 0, 1, \ldots\}$ with one step transition probability matrix $P = [P_{ij}], i, j \in S$ satisfying $\sum_j P_{ij} = \sum_i P_{ij} = 1$. Find the steady state distribution for this DTMC, if it exist.

(b) Consider the simple random walk on a circle. Assume that K odd number of points labeled $0, 1, \ldots, K-1$ are arranged on a circle clockwise. From i, the walker moves to $i+1$ (with K identified with 0) with probability p $(0 < p < 1)$ and to $i-1$ (with -1 identified with $K-1$) with probability $1-p$. Find the steady state distribution for this random walk, if it exist.

(3 + 2 marks)

8. Suppose the arrival at a counter form a time homogeneous Poisson process with parameter λ and suppose each arrival is of type A or of type B with respective probabilities p and $1-p$. Let $X(t)$ be the type of the last arrival before time t.

(a) Prove that $\{X(t), t \geq 0\}$ is a continuous time Markov chain.

(b) Find the steady state probabilities.

(3 + 2 marks)

9. In a parking lot with N (+ve integer) spaces the incoming traffic is according to a Poisson process with rate λ, but only as long as empty spaces are available. The occupancy times have an exponential distribution with mean $1/\mu$. Without loss of generality, assume that the system is modeled as a birth and death process. Let $X(t)$ be the number of occupied parking spaces at time t. Write the generator matrix Q. Write the forward Kolmogorov equations for the Markov process $\{X(t), t \geq 0\}$. Derive the equilibrium probability distribution of the process.

(2 + 2 + 2 marks)

10. Consider a $M/M/1$ queueing model.

(a) Find the waiting time distribution for any customer in this queueing model.

(b) Deduce the mean waiting time from the above distribution.

(c) Further, find the mean number of customers in the system, in a longer run.

(4 + 1 + 2 marks)