Department of Mathematics
MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Major Test (I Semester 2015 - 2016)

Time allowed: 2 hours Max. Marks: 50

1. (a) Let $\Omega = \{0, 1, 2, \ldots\}$. Let $\mathcal{F}$ be the collection of subsets of Ω that are either finite or whose complement is finite. Is $\mathcal{F}$ a σ-field? Justify your answer.

   (b) Consider a probability space $(\Omega, \mathcal{F}, P)$ with $\Omega = \{0, 1, 2\}$, $\mathcal{F} = \{\emptyset, \{0\}, \{1, 2\}, \Omega\}$, $P(\{0\}) = 0.5 = P(\{1, 2\})$. Give an example of a real-valued function on Ω that is NOT a random variable. Justify your answer. (2 + 3 marks)

2. Let X be uniformly distributed random variable over the interval $[0, 10]$. Find the CDF of $Y = \max\{2, \min\{4, X\}\}$. (4 marks)

3. For each fixed $\lambda > 0$, let X be a Poisson distributed random variable with parameter λ. Suppose λ itself is a random variable following a gamma distribution with pdf

$$f(\lambda) = \begin{cases} \frac{1}{\Gamma(n)}\lambda^{n-1}e^{-\lambda}, & \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$$

where n is a fixed positive constant. Find the pmf of the random variable X. (4 marks)

4. Pick the point (X, Y) uniformly in the triangle $\{(x, y) \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq x\}$. Calculate $E[(X - Y)^2/X]$. (4 marks)

5. In a communication system, the carrier signal at the receiver is modeled by $Y(t) = X(t)\cos(2\pi wt + \Theta)$ where $\{X(t), t \geq 0\}$ is a zero-mean and wide sense stationary process, Θ is a uniform distributed random variable with interval $(-\pi, \pi)$ and w is a positive constant. Assume that, Θ is independent of the process $\{X(t), t \geq 0\}$. Is $\{Y(t), t \geq 0\}$ wide sense stationary? Justify your answer.

(5 marks)

6. Consider a time-homogeneous discrete time Markov chain $\{X_n, n = 0, 1, \ldots\}$ with state space $S = \{0, 1, 2, 3, 4\}$ and one-step transition probability matrix $P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

   (a) Classify the states of the chain as transient, +ve recurrent or null recurrent.

   (b) When $P(X_0 = 2) = 1$, find the expected number of times the Markov chain visit state 1 before being absorbed.

   (c) When $P(X_0 = 1) = 1$, find the probability that the Markov chain absorbs in state 0.

(2 + 3 + 2 marks)

7. Consider a time-homogeneous continuous time Markov chain $\{X(t), t \geq 0\}$ which takes the value 0 and 1 with probability $\pi_0(t)$ and $\pi_1(t)$ at any time t, respectively. Also

$$Prob\{X(t + \Delta t) = 1 \mid X(t) = 0\} = \alpha\Delta t + o(\Delta t)$$

and

$$Prob\{X(t + \Delta t) = 0 \mid X(t) = 1\} = \beta\Delta t + o(\Delta t).$$

As $\Delta t \to 0$, $o(\Delta t) \to 0$. Assume that, α and β are positive constants. Assume $\pi_0(0) = 1$.

(a) Draw the state transition diagram for the the Markov chain $\{X(t), t \geq 0\}$.

(b) Write the Kolmogorov forward equations for the Markov chain $\{X(t), t \geq 0\}$.

(c) Derive the transient or time-dependent probability distribution of the Markov chain.

(2 + 2 + 3 marks)

8. (a) Define Poisson process.

(b) Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. For any $s, t \geq 0$, find $P(N(t+s) - N(t) = k/N(u); 0 \leq u \leq t)$

(c) Let $\{N(t), t \geq 0\}$ be a Poisson process with rate 5. Compute $P(N(2.5) = 15, N(3.7) = 21, N(4.3) = 21)$.

(3 + 2 + 2 marks)

9. Consider a $M/M/1$ queuing model with arrival rate λ and service rate μ.

(a) Derive the expression for π_n the steady state probability that n customers in the system.

(b) Find the average time spend in the queue by any customer.

(c) Find the service rate where customers arrive at a rate of 3 per minute, given that 95% of the time the queue contains less than 10 customers.

(3 + 2 + 2 marks)