33 + 1 = 34

MTL 106 (Introduction to Probability Theory and Stochastic Processes)

Time allowed: 2 hours Major Examination Max. Marks: 52

Name: ~~2016~~ AKSHAT AGARWAL

Entry Number: 2016EE30511 Signature:

Multiple Selection Questions: Section 1 **(6 × 2 = 12 marks)**

Each of the following questions 1 to 6 has four options out of which one or more options can be correct. Write A, B, C or D which corresponds to the correct option for the first correct answer followed by space and the next correct answer and so on. **2 marks** is awarded if all correct answers are written, **0 mark** for no answer or partial correct or any incorrect answer.

1. Let $(\Omega, \mathcal{F})$ be a sample space and $A \subset \Omega$ fixed. The function $X : \Omega \to \mathcal{R}$ is a random variable (RV). Which of the following statements are TRUE? largest σ-field

   (A) $X(w) = \begin{cases} 0, & w \in A \\ 1, & w \notin A \end{cases}$ (B) $X(w) = \begin{cases} 1, & w \in A \\ 0, & w \notin A \end{cases}$

   (C) $X(w) = \begin{cases} -1, & w \in A \\ 1, & w \notin A \end{cases}$ (D) $X(w) = \begin{cases} 0.5, & w \in A \\ 0.5, & w \notin A \end{cases}$ Answer: ABD (2 marks) 2

2. Let X be a non-negative integer valued random variable such that $E(X)$ exists. Which of the following statements are TRUE?

   (A) $E(X) = \sum_{k=0}^{\infty} P(X > k)$ (B) $E(X) = \sum_{k=0}^{\infty} P(X \geq k)$

   (C) $E(X) = \sum_{k=1}^{\infty} P(X > k)$ (D) $E(X) = \sum_{k=2}^{\infty} P(X > k)$ Answer: A (2 marks) 2

3. If $E[Y/X] = 1$, which of the following statements are NOT TRUE?

   (A) $Var[XY] \geq Var[X]$ (B) $Var[XY] > Var[X]$

   (C) $Var[XY] < Var[X]$ (D) $Var[XY] \leq Var[X]$ Answer: AB (2 marks)

4. Let $\{X_n, n = 0, 1, 2, \ldots\}$ be a DTMC with finite state space S and $i \in S$ is an absorbing state. Which of the following statements are TRUE? 2

   (A) The period of state i, $d_i = 1$. (B) The mean recurrence time of state i, $\mu_i = 1$.

   (C) State i is a +ve recurrent. (D) State i is a null recurrent. Answer: ABC (2 marks)

5. Let $N(t)$ be the random variable denoting the number of events occurs upto and including time t. Let $\{N(t), t \geq 0\}$ be a Poisson process with parameter λ. Let T_i be the inter-arrival times of the i^{th} event $i = 1, 2, \ldots$. Which of the following statements are TRUE? 2

   (A) $Cov(T_i, T_j) > 0$, for $i \neq j$ (B) $P(T_2 \leq t) = 1 - e^{-\lambda t}$, $0 \leq t < \infty$.

   (C) T_i's are independent. (D) $\sum_{i=1}^{n} T_i$ follows $B(n, \lambda)$. Answer: BC (2 marks)

6. Consider a $M/M/3/\infty$ queueing model where $X(t)$ denotes the number of customers in the system at any time t. Then, the number of customers undergoing service at time t is 2

   (A) $\min\{X(t), 3\}$ (B) $\max\{X(t), 3\}$ (C) 3 (D) $\min\{X(t), 1\}$ Answer: A (2 marks)

Comprehensive Type Questions: Section 2 $(2 + 2 + 4 + 4 \times 3 = 20$ **marks)**

Each of the following questions 7 to 13 has some subparts. For each subpart, write the answer upto 4 decimal places (D) or in fraction (F) or the expression (E). **1 mark/2 marks** is awarded if correct answer is written, **0 mark** for no answer or partial correct or any incorrect answer.

4

7. Let $P[X \le 0.29] = 0.75$, where X is a continuous type RV with some CDF defined over (0,1). If $Y = 1 - X$, Find k so that $P[Y \le k] = 0.25$. Answer(D): 0.71 (2 marks)

8. Let $0 < p < 1$ and N be a positive integer. Let $X \sim B(N, \frac{p}{N})$. Find $\lim_{N\to\infty} \left(1 - \frac{p}{N}\right)^N$, if it exists. Answer(E): e^{-p} (2 marks)

9. Let X_1 and X_2 be independent exponential distributed random variables with parameters 4 and 3 respectively. Define $X_{(1)} = \min\{X_1, X_2\}$ and $X_{(2)} = \max\{X_1, X_2\}$.

   (a) Find $Var(X_{(1)})$? Answer(F): (1 mark)

   (b) Find the distribution of $X_{(1)}$? Answer(E): exp(7/24) (1 mark)

   (c) Find $E(X_{(2)})$? Answer(F): 7/24 (2 marks)

10. Let X_0 be an integer-valued random variable, $P(X_0 = 0) = 1$, that is independent of the i.i.d. sequence $Z_1, Z_2, \dots$, where Z_n can take values in the set $\{-1, 1\}$ such that $P(Z_n = -1) = \frac{1}{3}, P(Z_n = 1) = \frac{2}{3}$. Let $X_n = X_{n-1} + Z_n$, $n = 1, 2, \dots$.

    (a) Find $P(X_3 = 1)$? Answer(F): 4/9 (1 mark)

    (b) Find the value of $P(X_5 = -1 \mid X_2 = 0)$? Answer(F): 2/9 (2 marks)

3

11. Let us assume that cars arrive according to a Poisson process at rate 8 per hour. Assume each car will pick up a hitchhiker (usually strangers, for a ride in their car) with probability $\frac{1}{8}$. You are third in line.

    (a) What is the probability that no cars arrive in first 30 minutes? Answer(E): e^{-4} (1 mark)

    (b) What is the probability that you will have to wait for more than 2 hours? Answer(E): $e^{-16} + \frac{16e^{-16}}{\lfloor 1} + \frac{(16)^2 e^{-16}}{\lfloor 2}$ (2 marks)

1

12. Assume the life times of $N = 200$ soldiers are iid following an exponential distribution with parameter μ, then the process of the number of surviving soldiers by time t, $\{X(t), t \ge 0\}$, is a pure death process with death rates $\mu_i = i\mu$, $i = 1, 2, \dots, N$. Assume that, $X(0) = N$.

    (a) Find $P(X(t) = N - 1)$? Answer(E): (2 marks)

    $200\mu e^{-200\mu t}$

    (b) Let S_N be the time of the death of the last member of the population, i.e., S_N is the time to extinction. Find $E(S_N)$? Answer(E): $\frac{1}{200\mu} + \frac{1}{199\mu} \cdots \frac{1}{\mu}$ (1 mark)

1

13. Patients visit a doctor in accordance with a Poisson process at the rate of 4 per hour, and the time doctor takes to examine any patient is independent exponential distributed with mean 6 minutes. All arriving patients attended by the doctor.

    (a) Write the Kendall notaion for the underlying queueing model. Answer(E): M/M/1/∞ (1 mark)

    (b) Find the expected waiting time (in minutes) in queue of any patient. Answer(F): 4 minutes (2 marks)

    $\frac{1}{\mu - \lambda} - \frac{1}{\mu}$

3

$200\mu e^{-200\mu t}$

2

~~$\frac{\rho^{n-1}}{\lfloor n-1} \Big/ 1 + \rho + \frac{\rho^2}{\lfloor 2} + \frac{\rho^3}{\lfloor 3} + \cdots + \frac{\rho^n}{\lfloor n}$~~

Scanned by CamScanner

Subjective Type Questions: **Section 3**

Write the answer in the same page provided for the questions 14 to 17. **Full marks** are awarded if all the steps are correct, and **partial marks** for an incorrect answer with wrong steps. $(4 \times 5 = 20$ marks$)$

14. Let $\Omega = \{1, 2, 3, 4\}$.

(a) Find three different σ-algebras $\{\mathcal{F}_n\}$ for $n = 1, 2, 3$ such that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$. (3 marks)

(b) Further, create a set function $P : \mathcal{F}_3 \to \mathcal{R}$ such that, $(\Omega, \mathcal{F}_3, P)$ is a probability space. (2 marks)

a) $\Omega = \{1,2,3,4\}$ $\quad F_1 \subset F_2 \subset F_3$

Consider $F_3 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{2,3,4\}, \{1,3,4\}, \{1,2,4\}, \phi, \Omega\}$ → largest σ field

$\boxed{F_2 = \{\phi, \{1\}, \{2,3,4\}, \Omega\}}$

& $\boxed{F_1 = \{\phi, \Omega\}}$

F_1, F_2 & F_3 are σ-algebra's & also $F_1 \subset F_2 \subset F_3$

3

b) $P: F_3 \to R$ such that (Ω, F_3, P) is a probability Space

~~Since F_3 is the largest σ field (as designed & defined above), any function on it would be a random varia~~

Let
$$\boxed{\begin{aligned} P\{X=1\} &= 1/4 \\ P\{X=2\} &= 1/4 \\ P\{X=3\} &= 1/4 \\ \& \ P\{X=4\} &= 1/4 \end{aligned}}$$

① $P\{X=i\} \geq 0$ for all $i = 1,2,3,4$

② $\sum_{i=1}^{4} P\{X=i\} = 1$ → Both these properties are satisfied

2

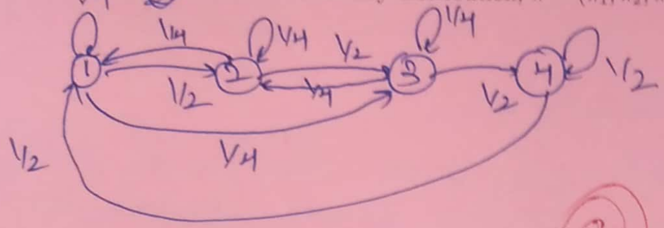
Required set function so that Ω, F_3, P is a probability space

18. Consider a DTMC $\{X_n, n = 0, 1, \ldots\}$ with $S = \{1, 2, 3, 4\}$ and its one-step transition probability matrix

$$P = \begin{pmatrix} 1/4 & 1/2 & 1/4 & 0 \\ 1/4 & 1/4 & 1/2 & 0 \\ 0 & 1/4 & 1/4 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$$

(a) Classify the states as transient, + recurrent or null recurrent. (2 marks)

(b) Find the stationary distribution, $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$, if it exists. (3 marks)



$f_{11} = f_{11}^{(1)} + f_{11}^{(2)} + f_{11}^{(3)} - - -$

$= \frac14 + (\frac14)(\frac12) + (\frac14)(\frac14)(\frac12) + [(\frac12)(\frac12)(\frac12)(\frac12) + (\frac14)(\frac12)(\frac14)(\frac14) + (\frac12)(\frac12)(\frac14)(\frac14)]$

$= 1$

2

Similarly, we can calculate $f_{22} = f_{33} = f_{44} = 1 \rightarrow$ and also since there are finite no of states & only one equivalent class $\rightarrow$ ALL STATES ARE POSITIVE RECURRENT

b) For stationary distribution, $\pi = \pi P$

$\Rightarrow (\pi_1 \ \pi_2 \ \pi_3 \ \pi_4) = (\pi_1 \ \pi_2 \ \pi_3 \ \pi_4) \begin{pmatrix} 1/4 & 1/2 & 1/4 & 0 \\ 1/4 & 1/4 & 1/2 & 0 \\ 0 & 1/4 & 1/4 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$

& also $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \rightarrow (1)$

3

$\Rightarrow \pi_1 = \frac{\pi_1}{4} + \frac{\pi_2}{4} + \frac{\pi_4}{2} \rightarrow (2)$

$\pi_2 = \frac{\pi_1}{2} + \frac{\pi_2}{4} + \frac{\pi_3}{4} \rightarrow (3)$

$\pi_3 = \frac{\pi_1}{4} + \frac{\pi_2}{2} + \frac{\pi_3}{4} \rightarrow (4)$

$\pi_4 = \frac{\pi_3}{2} + \frac{\pi_4}{2} \rightarrow (5)$

$\cancel{\frac{\pi_4}{4} + \frac{\pi_2}{4} =}$

$\frac{3\pi_1}{4} = \frac{\pi_2}{4} + \frac{\pi_3}{2}$

$\frac{3\pi_3}{4} = \frac{\pi_1}{4} + \frac{\pi_2}{2}$

& also $\pi_1 + \pi_2 = 1 - 2\pi_3$

$\cancel{\Rightarrow \frac{\pi_1}{2} = (1 - 2\pi_3) - \frac{3\pi_3}{2}}$

$\cancel{\Rightarrow \pi_1 = 2 - 4\pi_3 - 3\pi_3 = 2 - 7\pi_3}$

$\cancel{\pi_2 = 1 - 2\pi_3 - (2 - 7\pi_3) = 5\pi_3 - 1}$

$\Rightarrow \frac{\pi_4}{2} = \frac{\pi_3}{2}$ or $\pi_4 = \pi_3$

$\Rightarrow \pi_1 = \pi_2 = \pi_3 = \pi_4 = \frac14$

$\boxed{\pi = (\frac14, \frac14, \frac14, \frac14)}$

10. Let $\{X(t), t \geq 0\}$ be a Poisson process with parameter λ and $X(0) = j$ where j is a positive integer. Consider the random variable $T_j = \inf\{t : X(t) = j + 1\}$, i.e., T_j is the time of occurrence of the first jump after the jth jump, $j = 1, 2, \ldots$.

(a) Find the distribution of T_1. (2 marks)

(b) Find the joint distribution of $(T_{2017}, T_{2018}, T_{2019})$. (3 marks)

a) $X(0) = J$ $\{X(t), t \geq 0\}$

$T_J = \inf\{t : X(t) = J+1\}$

0

$T_1 = \inf\{t : X(t) = 2\} \rightarrow$ Distribution of T_1 = poisson distribution with parameter $(2\lambda t)$

b) Joint distribution of $(T_{2017}, T_{2018}, T_{2019})$

0

$T_{2017} = \inf\{t : X(t) = 2018\} \rightarrow$ poisson distribution with parameter $(2018\lambda t)$

$T_{2018} = \inf\{t : X(t) = 2019\} \rightarrow$ poisson " " " $2019\lambda t$

$T_{2019} = \inf\{t : X(t) = 2020\} \rightarrow$ " " " $2020\lambda t$

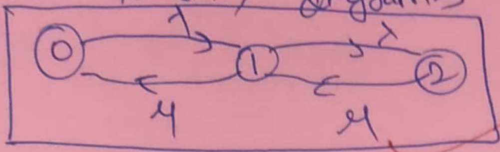
$\therefore$ Joint distribution of $T_{2017}, T_{2018}, T_{2019} \rightarrow$ product of the individual poisson distributions with respective parameters for all 3.

17. Consider a $M/M/1/2$ queuing system. Let $X(t)$ be a random variable denoting the number of customers in the system at any time t.

(a) Draw the state transition diagram for the process $\{X(t), t \geq 0\}$. (1 mark)

(b) Let $\pi_n(t)$ be the probability that there are n customers in the system at time t given that there was no customers at time 0. Find the time-dependent probabilities $\pi_n(t)$ for $n = 0, 1, 2$. (4 marks)

a) State transition diagram →



1

M/M/1/2

No. of servers = 1

& Max. capacity = 2

b) Generator Matrix Q is given by →

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & \mu & -\mu \end{pmatrix}$$

$\pi_n(t)$ = probability that there are n customers at time t given there was no customer at time 0

$= (\pi_1, \pi_2, \pi_3)$

$\Rightarrow \pi Q = 0 \Rightarrow (\pi_1 \;\; \pi_2 \;\; \pi_3) \begin{pmatrix} -\lambda & \lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & \mu & -\mu \end{pmatrix} = 0$

& also $\pi_1 + \pi_2 + \pi_3 = 1 \rightarrow (4)$

$\Rightarrow -\lambda\pi_1 + \mu\pi_2 = \pi_1 \rightarrow (1)$

$+\lambda\pi_1 - (\lambda+\mu)\pi_2 + \mu\pi_3 = \pi_2 \rightarrow (2)$

$\lambda\pi_2 - \mu\pi_3 = \pi_3 \rightarrow (3)$

On solving (1), (2), (3) & (4) → we get

$$\boxed{\pi_1 = \frac{1}{1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2}}} \quad \boxed{\pi_2 = \frac{\lambda/\mu}{1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2}}} \quad \& \quad \boxed{\pi_3 = \frac{\lambda^2/\mu^2}{1 + \lambda/\mu + \lambda^2/\mu^2}}$$

→ Required time dependent probabilities

1