Department of Mathematics

MTL 106 (Probability and Stochastic Processes)

Major Exam

Time: 2 hour 15 minutes **Date: 08/01/2021**
Max. Marks: 50

Note: The exam is closed-book, and all the questions are compulsory.

1. Let $\{X_n\}$ be a sequence of **i.i.d** random variables, defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with uniform distribution on $(-1, 1)$. Let

$$Y_n = \frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} X_i^2 + X_i^3}.$$

Show that $\sqrt{n}Y_n$ converges in distribution as $n \to \infty$. Let $\phi_n(t)$ be the characteristic function of $\sqrt{n}Y_n$. Calculate $\lim_{n\to\infty} \phi_n(2)$.

([5+2])

2. Let $\{X_n : n \geq 0\}$ be a discrete-time Markov chain (DTMC) with state space $S = \{1, 2, 3\}$ and transition probability matrix $P = \begin{pmatrix} 0.2 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.2 \\ 0.4 & 0.6 & 0 \end{pmatrix}$.

a) Check whether the chain is ergodic or not, and find $p_{31}^{(2)}$.

b) Examine whether there exists a stationary distribution of the given DTMC or not.

c) Calculate $p_{13}^{(n)}$ for large n.

[(2+1)+2+4]

3. Arrivals of customers into a store follow a Poisson process with arrival rate (intensity) 20 per hour. Suppose that the probability of a customer buys something is $p = 0.4$.

a) What is the probability that no sales are made in a first 10 minutes period?

b) Find the expected number of sales made during an eight-hours business day.

c) What is the probability that 25 sales are made in first 1.5 hours given that 40 sales are made in 3 hours from the opening of the store.

d) Find the probability that 20 or more sales are made in two hours.

[1+2+3+2]

4. Let $\{X(t) : t \geq 0\}$ be a continuous-time Markov chain (CTMC) with finite state space $S = \{1, 2, 3\}$, transition rate matrix $Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix}$ and initial distribution $\lambda = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$.

a) Find the transition probability matrix P of **embedded Markov chain**.

b) Let $P(t) = \big(p_{ij}(t)\big)_{i,j\in S}$ be the transition probability matrix of the given CTMC. Show that, for all $i \in S$,

$$p'_{1i}(t) = -2p_{1i}(t) + p_{2i}(t) + p_{3i}(t),$$
$$p'_{3i}(t) = p_{1i}(t) + 2p_{2i}(t) - 3p_{3i}(t)\,.$$

c) Calculate $\lim_{t\to\infty} p_{i2}(t)$ for all $i \in S$.

[1+2+3]

5. Let $\{X(t) : t \geq 0\}$ be a birth and death process with birth and death rate λ_n resp. μ_n where

$$\lambda_n = 2n+1, \quad n \geq 0$$
$$\mu_n = 3n, \quad n \geq 1\,.$$

Write down Kolmogorov's forward equation for $\{X(t) : t \geq 0\}$. Find its stationary distribution.

[2+4]

6. The time spent by Peter on his phone every day is a random variable X which follows a normal distribution with mean 28 minutes and standard deviation 8 minutes. It is known that the time spent by Peter on different days are independent.

- On a given day find the probability that Peter uses his phone for i) less than 30 minutes, ii) between 10 and 20 minutes.
- Calculate an interval, symmetrical about 28 minutes, within which X lies with on 0.8 probability.
- Peter uses his phone for n days. For what values of n, with at least 0.25 probability the mean time spent by Peter on his phone is at least 30 minutes

[2+2+3]

7. (*a*) Suppose an experiment having r possible outcomes $1, 2, 3, \ldots, r$ that occur with probabilities $p_1, p_2, \ldots, p_r$ is repeated n times. Let X be the number of times the first outcome occurs, and let Y be the number of times the second outcome occurs. Then, compute the correlation coefficient $\rho(X,Y)$.

(*b*) Consider a random variable with PMF given by

$$P\{X = x\} = \begin{cases} \frac{1}{18}, & x = 1,3 \\ \frac{16}{18}, & x = 2 \end{cases}$$

Show that there exists k for which the general bound given by Chebyshev's inequality cannot be improved; k denotes the constant used in Chebyshev's inequality.

[4+3]