$1-2\sin^2\left(\frac{\theta}{2}\right) = \cos\theta$

$1-2\sin^2(\theta) = \cos(2\theta)$

$|a|+|b|+|c| \geq |a+b+c|$

$f(y|X=x)$

$\Rightarrow \begin{cases} \lambda e^{-\lambda y} \end{cases}$

Department of Mathematics, IIT Delhi

2202-MTL100: Major Exam.

Time: 2 hours **Date:** 06-05-2023 **Total Marks:** 50

"As a student of IIT Delhi, I will not give or receive aid in examinations. I will do my share and take an active part in seeing to it that others as well as myself uphold the spirit and letter of the Honour Code."

$p(1-p)^k$

Q.1) (i) Suppose X follows a uniform distribution on $(1,2)$ and given $X = x$, Y is an exponential distribution with rate parameter x. Find $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$.

(ii) Let X and Y be two iid random variables which follow uniform distribution on $(0,1)$. Let $Z = \max(X,Y)$ and $W = \min(X,Y)$. Find $\mathrm{Cov}(Z,W)$.

(iii) Let X follows a geometric distribution with parameter p and let $M > 0$ be an integer. Define $Y = \max\{X, M\}$ and $Z = \min\{X, M\}$. Find $\mathbb{E}(Y)$ and $\mathbb{E}(Z)$.

(iv) Let N be the number of phone calls made by the customers of a phone company in a given hour. Suppose that N follows a Poisson distribution with parameter $\beta > 0$. Let X_i be the length of the ith phone call for $i = 1,2,\ldots,N$. We assume that X_i's are independent of each other and also independent of N. The PDF of each X_i is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

$\int_3^8 x f(x)\,dx$

Let Y be the sum of the lengths of all the phone calls. Find $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$.

(v) Let X follows a binomial distribution where n are the number of trials and success probability is $\frac{1}{2}$. Show that $\mathbb{P}(X \geq \frac{3n}{4}) \leq \frac{4}{n}$.

(vi) Let X be a normal distribution with parameters μ and $\sigma^2 = 0.25$. Find a constant c such that $\mathbb{P}(|X-\mu| \leq c) = 0.9$. ($\phi(1.64) = 0.95$, where $\phi(x)$ is standard normal CDF)

2+3+3+1+1+1 marks

Q.2) (a) Let X, Y be i.i.d RVs with common PDF

$u+v \geq 0$

$u > 0$

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Let $U = X+Y$ and $V = X-Y$. Find the conditional PDF of V given $U = u$, for some fixed $u > 0$. What is $\mathbb{E}(V|U)$?

b) Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d random variables with common distribution function $F(\cdot)$ and finite mean and variance. Define $G_n := \sum_{i=1}^{n} \mathbf{1}_{\{X_i \leq 3\}}$.
Calculate the following limit:

$$\lim_{n\to\infty} \mathbb{E}\left[\sin^2\left(\frac{G_n - nF(3)}{\sqrt{n}}\right)\right].$$

(3+1)+5 marks

$\mu = \frac{n}{2}$ $\quad p\left(X \geq \frac{n}{2} + \frac{n}{4}\right)$ $\quad \sigma^2 = \frac{n}{4}$

$p\left(X - \frac{n}{2} \geq \frac{n}{4}\right)$ $\quad \dfrac{\left(\frac{n}{4}\right)}{\left(\frac{n}{4}\right)^2}$

/20

$\pi = \pi P$

$\pi P = (\pi P)P \qquad \pi P = P$

Q.3) Consider a DTMC $\{X_n : n \geq 0\}$ with finite state space $S = \{1, 2, 3\}$ and transition probability matrix $P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 \end{pmatrix}$.

a) Show that for any row vector v, $\|vP\|_1 \leq \|v\|_1$, where for any row vector $w = (w_1, w_2, w_3)$, $\|w\|_1 := \sum_{i=1}^{3} |w_i|$.

b) We say that $\lambda \in \mathbb{R}$ is a left eigen-value of P, if there exists a non-zero row vector v such that $vP = \lambda v$. Such v is called left eigen-vector of P corresponding to the left eigen-value λ. Deduce that all left eigen-value λ of P must satisfies $|\lambda| \leq 1$.

c) Explain whether 1 is a left eigen-value of P or not.

d) Calculate $\lim_{n\to\infty} p_{i2}^{(n)}$ for all $i \in S$, where $P^{(n)} = (p_{ij}^{(n)})_{i,j\in S}$ is the nth step transition probability matrix.

3+1+2+4 marks

Q.4) a) Assume that the passengers arrive at a bus station as a Poisson process with rate 5. The only bus departs after a (deterministic) time T. Assume that the arrival time of passengers are independent and uniformly distributed over $[0, T]$. Find the expected total waiting time for all passengers.

b) Arrivals of customers into a super market follow a Poisson process with arrival rate 20 per hour. Suppose that the probability of a customer buys something is 0.8.

i) Find the expected number of sales made during an eight hours business day.

ii) What is the probability that 25 sales are made in first 2 hours given that 55 sales are made in 4 hours from the opening of the store.

c) Let $\{N(t) : t \geq 0\}$ be a Poisson process with intensity 3. Prove that

$$\frac{1}{3t+1} \leq \mathbb{P}(N(t) < 3t + 1) \leq 1, \quad \forall\, t \geq 0.$$

5+(1+2)+2 marks

Q.5) a) Let $\{X(t) : t \geq 0\}$ be a birth and death process with birth and death rate λ_n and μ_n respectively, where $\lambda_n = n + 2$, $n \geq 0$, $\mu_n = 2n$, $n \geq 1$. Show that the second order moment $m_2(t)$ around zero of $X(t)$ satisfies the differential equation:

$$m_2'(t) = 7m(t) - 2m_2(t) + 2,$$

where $m(t)$ is the mean population size at time t.

b) At petrol pump, customers arrive according to a Poisson process with an average time of 6 minutes between arrivals. The service time is exponentially distributed with mean time = 3 minutes.

i) What is the average time spent by a car in the petrol pump?

ii) What is the average waiting time of a car before receiving petrol ?

iii) Prove or disprove: a car waits more than 10 minutes with probability $\frac{1}{2}e^{-\frac{5}{3}}$.

5+(1+1+3) marks

——————— **Best of Luck!!!** ———————

10

$\frac{1}{6} \times 10$

$1 - \rho e^{-\rho(\lambda - \mu)x}$

$1 - \frac{1}{2} e^{-\frac{1}{2}(\frac{1}{6} - \frac{1}{3})x}$