MTL 106 (Introduction to Probability Theory and Stochastic Processes)

Time allowed: 1 hour Minor 1 Examination Max. Marks: 25

Name: Entry Number: Signature:

Multiple Selection Questions: **Section 1** ($1 \times 5 = 5$ **marks**)

Each of the following questions 1 and 2 has four options out of which one or more options can be correct. Write A, B, C or D which corresponds to the correct option for the first correct answer followed by space and the next correct answer and so on. **1 mark** is awarded if all correct answers are written, **0 mark** for no answer or partial correct answers or any incorrect answer.

1. Let $\Omega = \{a, b, c, d\}$. Which of the following sets are NOT σ-fields on Ω.
(A) $\{\emptyset, \{a\}, \{b\}, \{c, d\}, \Omega\}$ (B) $\{\emptyset, \{a\}, \{b, c, d\}, \Omega\}$
(C) $\{\emptyset, \{a, b, d\}, \{c, d\}, \Omega\}$ (D) $\{\emptyset, \{b\}, \{a, c, d\}, \Omega\}$ Answer: A, C

2. Let X be uniform distributed random variable on the interval $(0, 1)$. Suppose $Y = g(X)$ is uniform distributed random variable on the interval (a, b) with $-\infty < a < b < \infty$. Then, $g(X)$ is (A) $aX + (b - a)$ (B) $a + (b - a)X$ (C) $bX + (a - b)$ (D) $b + (a - b)X$
Answer: B, D

3. The probability generating function for $B(n, p)$ is
(A) $(qt+p)^n$ (B) $(pt+1-p)^n$ (C) $(pt+p)^n$ (D) $(qt+1-p)^n$ Answer: B

4. Which of the following distributions NOT satisfy the memoryless property?
(A) Poisson (B) Bernoulli (C) Geometric (D) Exponential Answer: A, B

5. If X is a random variable on a measurable space $(\Omega, \mathcal{F})$ where $\mathcal{F}$ is the largest σ-field on Ω, then which of the following real-valued functions will be random variables on this measurable space $(\Omega, \mathcal{F})$. (A) X^- (B) $| X |$ (C) X^2 (D) X^+ Answer: A, B, C, D

——————Rough Work——————

Numeric Type Questions: **Section 2** **($5 \times 2 = 10$ marks)**

Write the answer upto 4 decimal places (D) or in fraction (F) or the expression (E) for the following questions 6 to 10. **2 marks** are awarded if answer is correct, and **0 mark** for no answer or an incorrect answer.

6. Let $\Omega = \{s_1, s_2, s_3, s_4\}$ and $P\{s_1\} = \frac{1}{5}$, $P\{s_2\} = \frac{1}{10}$, $P\{s_3\} = \frac{2}{5}$, $P\{s_4\} = \frac{3}{10}$. Define, $A_n = \begin{cases} \{s_1, s_3\} & \text{if } n \text{ is odd} \\ \{s_2, s_4\} & \text{if } n \text{ is even} \end{cases}$. Find $P(\limsup A_n)$. Answer(F/D): 1

7. Suppose that the number of passengers for a limousine pickup is thought to be either 1, 2, 3, or 4, each with equal probability, and the number of pieces of luggage of each passenger is thought to be 1 or 2, with equal probability, independently for different passengers. What is the probability that there will be five or more pieces of luggage? Answer(F/D): $23/64$

8. Suppose X has a geometric distribution with parameter p. Then, the moment generating function for X is given by: Answer(E): $pe^t/(1-qe^t)$

9. Let $X \sim \mathrm{P}(\lambda)$ such that $P(X = 0) = e^{-3}$. Find $Var(X)$. Answer(E): 3

10. Let X be a normal distributed random variable with mean 0 and variance 9. Then, the pdf of $\cancel{h(X)}$ is given by:

$\mu = 0$ $\quad \sigma^2 = 9$

Answer(E):

$Y = e^X$

$$f_Y(y) = \begin{cases} \frac{1}{y\sigma\sqrt{2\pi}} e^{-\left(\frac{\log y - \mu}{\sigma\sqrt{2}}\right)^2} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Rough Work

Subjective Type Questions: **Section 3** **($2 \times 5 = 10$ marks)**

Write the answer in the same page provided for the questions 11 and 12. **Full marks** are awarded if all the steps are correct, and **partial marks** for an incorrect answer with wrong steps.

11. (a) Write the Kolmogorov axiomatic definition of probability. (2 marks)

(b) Let $\Omega = \Re$ and $\mathcal{F}$ be the Borel σ-field on $\Re$. For each interval $I \subseteq \Re$ with end points c and d ($c \leq d$), let

$$P(I) = \int_c^d \frac{1}{\pi} \frac{1}{1+x^2} dx \ .$$

Does P define a probability on the measurable space $(\Omega, \mathcal{F})$? Justify your answer. (3 marks)

Answer: (a) Let E be a random experiment, Ω be the collection of all possible outcomes of E and $\mathcal{F}$ be the σ-field on Ω.

The function P defined on $\mathcal{F}$ such that

(i) $P(A) \geq 0 \ \forall A \in \mathcal{F}$] - 1

(ii) $P(\Omega) = 1$

(iii) If A_i's are mutually exclusive events in $\mathcal{F}$ then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ - 1

Then P is called a probability function.

b) (i) For any interval I

$$P(I) = \int_c^d \frac{1}{\pi} \frac{1}{1+x^2} dx = \frac{1}{\pi}(\tan^{-1} d - \tan^{-1} c)$$

since $\tan^{-1}(\ )$ is monotonically increasing and $d \geq c$ — 1

$\therefore P(I) \geq 0$

(ii) $P(\Omega) = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1+x^2} dx = \frac{1}{\pi}(\tan^{-1}\infty - \tan^{-1}-\infty) = 1$ — 1

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(iii) for disjoint intervals $I_1, I_2 - - -$

$$P\left(\bigcup_{i=1}^{\infty} I_i\right) = \int_{\bigcup_{i=1}^{\infty} I_i} \frac{1}{\pi} \frac{1}{1+x^2} dx = \int_{I_1} \frac{1}{\pi(1+x^2)} dx + - - - - + - - = \sum_{i=1}^{\infty} P(I_i)$$ — 1

12. (a) Let X be a continuous type random variable with pdf

$$f(x) = \begin{cases} \beta + \alpha x^2, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

If $E(X) = 3/5$, find the value of α and β. (2 marks)

(b) Prove that, for any random variable X, $E(X^2) \geq [E(X)]^2$. When does one have equality? (2 + 1 marks)

Answer: a.) $E(X) = 3/5 \Rightarrow \int_0^1 x(\beta + \alpha x^2)\, dx = 3/5$

$\Rightarrow \beta \frac{x^2}{2} + \alpha \frac{x^4}{4} \Big|_0^1 = 3/5$

Also, $\int_0^1 f(x) dx = 1 \Rightarrow \int_0^1 \beta + \alpha x^2 \; dx = 1$

$\Rightarrow \beta x + \frac{\alpha x^3}{3} \Big|_0^1 = 1 \Rightarrow \frac{\beta}{1} + \frac{\alpha}{3} = 1$ - 1

$\Rightarrow \beta = 1 - \frac{\alpha}{3}$ and $\frac{\beta}{2} + \frac{\alpha}{4} = 3/5 \Rightarrow \alpha = 6/5$ - 1

and $\beta = 3/5$

b) 1.) $\text{var } X = E[(X - \mu)^2]$

Since $(X - \mu)^2 \geq 0$ for all possible values of X 1

$\Rightarrow \text{var } X \geq 0$ -

2.) $\text{var } X = E(X^2) - (E(X))^2 \geq 0$

$\Rightarrow E(X^2) \geq (E(X))^2$ - 1

3) For equality $E(X^2) = (E(X))^2$

$\Rightarrow X$ is a degenerate random variable ie. 1

$P(X = k) = 1$ for some constant $k \in \mathbb{R}$.