Department of Mathematics

MTL 106 (Probability and Stochastic Processes)

Minor Examination

Time: 1 hour **Date: 22/09/2021**
Max. Marks: 30

Note: The exam is closed-book, and all the questions are compulsory.

Q.1 The following questions can have multiple correct answers. Write all the correct answers. The marks will be awarded only if you write all correct answers.

(a) Let the random variables X and Y defined on sample space $(\Omega, \mathcal{F})$ have same PMF. Then,

(i) $X(\omega) = Y(\omega)$, $\forall\ \omega \in \Omega$, (ii) CDF of X and Y are same, (iii) Characteristic function of X and Y are same.

(b) Let the random variables X and Y are such that $E(XY) = E(X)E(Y)$. Then,

(i) X and Y are independent , (ii) Cov(X, Y)= 0, iii) Cov$(X - Y, Y)$ =Var(Y).

(c) Let $X : \Omega \to \mathbb{R}$ be a random variable defined on probability space $(\Omega, \mathcal{F}, P)$. Let Q be a probability distribution of X and $\mathcal{B}$ is Borel σ-field. Then,

(i) $Q : \mathbb{R} \to [0, 1]$ (ii) $Q : \mathcal{B} \to [0, 1]$ (iii) $Q((-\infty, x)) = P\{X \leq x\}$ (iv) Q is continuous.

(d) Let X_1, X_2, X_3, X_4 be pairwise independent random variables and g_1, g_2 are Borel measurable functions. Then,
(i) X_1, X_2, X_3 are independent (ii) $g(X_1)$ and $g(X_2)$ are independent (iii) $g(X_1)$, $g(X_2)$, $g(X_3)$ are independent (iv) X_1 and X_2 are independent.

(e) Let the CDF of random variable X is given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{2} & \text{if } 0 \leq x < 1 \\ \frac{2}{3} & \text{if } 1 \leq x < 2 \\ \frac{11}{12} & \text{if } 2 \leq x < 3 \\ 1 & \text{if } 3 \leq x. \end{cases}$$

Then, (i) X is a discrete random variable, (ii) $P\{X = 1\} = \frac{1}{6}$, (iii) $P\{X = 1\}$ cannot be found from given information, (iv) $P\{X \leq 1\} = \frac{1}{2}$.

(1+1+1+1+1 marks)

Q.2 Give the final answer to the following questions. The justification of the answers is not required. The step marking is not applicable in these questions.

(a) Let X be a continuous random variable with PDF

$$f(x) = \begin{cases} \alpha + \beta x^2, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

If $E(X) = \frac{3}{4}$, find the values of α and β.

(2 marks)

(b) Let X be a discrete random variable with MGF $M_X(t) = \alpha + \beta e^{4t}$, $E(X) = 3$. Find i) α, β, ii) PMF of X.

(1+1 marks)

(c) A total of 4 buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying this randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on the bus of randomly selected driver. Compute $E[X]$ and $E[Y]$.

(2 marks)

(d) A fair coin is tossed three times. Let X = number of heads in three tossings, and Y = difference, in absolute value, between number of heads and number of tails. What is conditional PMF of X given $Y = 1$.

(2 marks)

(e) Consider an experiment having three possible outcomes, 1, 2, 3, that occur with probabilities 1/3, 1/3, and 1/3, respectively. Suppose two independent repetitions of the experiment are made and let X_i, $i = 1, 2, 3$, denote the number of times the outcome i occurs. What is the PMF of $X_1 + X_2$?

(2 marks)

Q.3 The following questions are descriptive type. Please provide detailed answers.

1. Let X, Y be iid RVs with common PDF

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Let $U = X + Y$ and $V = X - Y$. Find,
i) Joint PDF of U and V, ii) Marginals of U and V, iii) conditional PDF of V given $U = u$, for some fixed $u > 0$.

(8 marks)

2. Suppose that two buses, A and B, operate on a route. A person arrives at a certain bus stop on this route at time 0. Let X and Y be the arrival times of buses A and B, respectively, at this bus stop. Suppose that X and Y are independent random variables and density function of X is given by

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in [0, 2] \\ 0, & \text{otherwise,} \end{cases}$$

and PDF of Y is given by

$$f(y) = \begin{cases} \frac{1}{3}, & \text{if } y \in [0, 3] \\ 0, & \text{otherwise.} \end{cases}$$

What is the probability that bus A will arrive before bus B?

(3 marks)

3. Let X be a positive RV of the continuous type with PDF $f(\cdot)$. Find the PDF of the RV $U = \frac{X}{(1+X)}$. If, in particular, X has the PDF

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq x \leq 2, \\ 0, & \text{otherwise,} \end{cases}$$

what is the PDF of U?

(4 marks)