MTL 106 (Introduction to Probability Theory and Stochastic Processes)

Time allowed: 1 hour **Minor 2 Examination** **Max. Marks: ~~28~~ 24**

Name: Entry Number: Signature:

Multiple Selection Questions: **Section 1** ($1+2+2=5$ **marks**)

Each of the following questions 1, 2 and 3 has four options out of which one or more options can be correct. Write A, B, C or D which corresponds to the correct option for the first correct answer followed by space and the next correct answer and so on. **1 mark/2 marks** is awarded if all correct answers are written, **0 mark** for no answer or partial correct or any incorrect answer.

1. Suppose X has a uniform distribution on the interval $(-\pi, \pi)$. When $Y = \cos(X)$, the value of $\text{cov}(X, Y)$ is (A) 1 (B) 0 (C) -1 (D) 0.5 Answer: (1 mark)

2. Suppose that X has exponential distribution with the parameter λ. Which of the following statements are NOT TRUE?
(A) $E(X-4/X>6) = 1+\frac{1}{\lambda}$ (B) $E(X-2/X>3) = 1+\frac{1}{\lambda}$
(C) $E(X-3/X>3) = 1+\frac{1}{\lambda}$ (D) $E(X-3/X>3) = \frac{1}{\lambda}$ Answer: (2 marks)

3. Consider a DTMC $\{X_n, n = 0, 1, 2, \ldots\}$ with state space $S = \{0, 1\}$ and one-step transition probability matrix $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Which of the following statements are NOT TRUE?
(A) P is a doubly stochastic matrix. (B) P is a Markov matrix.
(C) P is a stochastic matrix. (D) P is a stationary matrix. Answer: (2 marks)

Numeric Type Questions: **Section 2** ($6 \times 2 = 12$ **marks**)

Write the answer upto 4 decimal places or in fraction or the expression for the questions 4 to 9. **2 marks** are awarded if answer is correct, and **0 mark** for no answer or an incorrect answer.

4. Let X and Y be independent random variables with $X \sim B(2, \frac{1}{3})$ and $Y \sim B(3, \frac{1}{2})$. Find $P(X = Y)$. Answer: 19/72 , For $X \sim B(3,\frac{1}{2})$, $Y \sim B(2, 1/3)$ Ans : 19/72 (2 marks)
For $X \sim B(3,\frac{1}{3})$, $Y \sim B(2,\frac{1}{2})$ and $X \sim B(2,\frac{1}{2})$, $Y \sim B(3,\frac{1}{3})$ Ans : 19/54

5. Amit and Supriya work independently on a problem in Tutorial Sheet 5 of Probability and Stochastic Processes course. The time for Amit to complete the problem is exponential distributed with mean 5 minutes. The time for Supriya to complete the problem is exponential distributed with mean 4 minutes. Given that Amit requires more than 1 minutes, what is the probability that he finishes the problem before Supriya? Answer: $\frac{4}{9}e^{-1/4}$ (2 marks)

6. Let X and Y be iid random variables, each having uniform distribution in the interval $(1, 2)$. Find $E\left(\frac{3X}{2Y}\right)$. Answer: $\frac{9}{4}\ln(2)$, $E\left(\frac{3X}{4Y}\right) = \frac{9}{8}\ln(2)$, $E\left(\frac{2X}{3Y}\right) = \ln(2)$ (2 marks)
$E\left(\frac{4X}{3Y}\right) = 2\ln(2)$

7. Consider polling of n voters and record the fraction S_n of those polled who are in favour of a particular candidate. If p is the fraction of the entire voter population that supports this candidate, then $S_n = \frac{X_1+X_2+\ldots+X_n}{n}$, where X_i are i.i.d. random variables with $B(1, p)$. How many voters should be sampled so that we wish our estimate S_n to be within 0.02 (α) of p with probability (Prob) at least 0.90? Answer: 6250 , when $\alpha = 0.02$ (2 marks)
& Prob = 0.95
$n \geq 12500$,
when $\alpha = 0.01$
& Prob = 0.90
$n \geq 25000$
when $\alpha = 0.01$
& prob = 0.95
$n = 50{,}000$

Q5

Amit mean time	Supriya mean time	Ans
3	4	$\frac{4}{7}e^{-\frac{1}{4}}$
4	4	$\frac{1}{2}e^{-\frac{1}{4}}$
4	3	$\frac{3}{7}e^{-\frac{1}{3}}$

Q9 For $A=240$, $n=60$, Prob $= 1-\phi\left(\frac{30}{\sqrt{175}}\right)$, For $A=280$, $n=60$ Prob $= 1-\phi\left(\frac{70}{\sqrt{175}}\right)$

For $A=225$, $n=50$, Prob $= 1-\phi\left(\frac{50}{\sqrt{\frac{875}{6}}}\right)$.

8. Consider Bacteria reproduction by cell division. In any time t, a bacterium will either die (with probability 0.2), stay the same (with probability 0.5), or split into 2 parts (with probability 0.3). Assume bacteria act independently and identically irrespective of the time. Given that there are 500 bacteria in the population at time $t = 200$, what is the expected number of bacteria at time $t = 201$. Answer: 550, For 1000 bacteria & $t=200$ Ans: 1100 (2 marks)
For 500 bacteria, $t=100$, Ans: 550, For 500 bacteria, $t=50$, Ans: 550

9. A person puts few one rupee coins into a piggy-bank each day. The number of one rupee coins added on any given day is equally likely to be 1, 2, 3, 4, 5 or 6, and is independent from day to day. Find an approximation to the probability that it takes at least to collect $\underset{A}{\underline{260}}$ rupees in $\underset{n}{\underline{60}}$ days? Final answer can be in terms of $\phi(z)$ where $\phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{t^2}{2}}dt$.
Answer: $1-\phi\left(\frac{50}{\sqrt{175}}\right)$ (2 marks)

Comprehensive Type Questions: **Section 3** (3 × 3 = 9 **marks)**

Each of the following questions 10 to 12 has some subparts. For each subpart, write the answer upto 4 decimal places or in fraction or the expression. **1 mark/2 marks** is awarded if correct answer is written, **0 mark** for no answer or partial correct or any incorrect answer.

10. Let X_1 and X_2 be two continuous type random variables with the joint pdf

$$f(x_1, x_2) = \begin{cases} Kx_1x_2, & 0 < x_1 < 1, 0 < x_2 < 1 \\ 0, & \text{otherwise} \end{cases}$$

   (a) Find the value of K? Answer: 4 (1 mark)

   (b) Define $Y_1 = X_1^2$ and $Y_2 = X_1X_2$. The joint pdf of (Y_1, Y_2) is given by:

$$\text{Answer: } f(y_1, y_2) = \begin{cases} \frac{2y_2}{y_1}, & 0 < y_2 < \sqrt{y_1} < 1 \\ 0, & \text{otherwise} \end{cases}$$ (2 marks)

when $Y_1 = X_1X_2$, $Y_2 = X_1^2$
$f(y_1, y_2) = \frac{2y_1}{y_2}$, $0 < y_1 < \sqrt{y_2} < 1$

11. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with $E(X_1) = \mu$ and $Var(X_1) = \sigma^2$. Define

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

   (a) Find $Var(\overline{X})$ for $n = 60$, $\mu = 100$ and $\sigma^2 = 3600$? Answer: $\sigma^2/n = 60$ (1 mark)

   (b) Find $E[S^2]$ for $n = 60$, $\mu = 100$ and $\sigma^2 = 3600$? Answer: $\sigma^2 = 3600$ (2 marks)

12. Let $\{X_n\}$ be a sequence of independent random variables defined by

$$P\{X_n = 0\} = 1 - \frac{1}{n}, \quad \text{and} \quad P\{X_n = 1\} = \frac{1}{n}, \quad n = 1, 2, \ldots.$$

   (a) ~~Find the distribution of X such that $X_n \xrightarrow{a.s} X$~~
   ~~Answer:~~ ~~(2 marks)~~

   (b) Find the distribution of X such that $X_n \xrightarrow{p} X$ $P(X=0) = 1$
   Answer: $P(X=1) = 0$ (1 mark)

when $P\{X_n = 0\} = \frac{1}{n}$ & $P(X_n = 1) = 1 - \frac{1}{n}$

Ans: $P(X=0) = 0$, $P(X=1) = 1$.

Q12 $n=50$, $\mu=100$, $\sigma^2=2500$ $Var(\overline{X}) = 50$, $E(S^2) = 2500$
$n=30$, $\mu=100$, $\sigma^2=900$ $Var(\overline{X}) = 30$, $E(S^2) = 900$
$n=40$, $\mu=100$, $\sigma^2=1600$ $Var(\overline{X}) = 40$, $E(S^2) = 1600$