MTL 106 (Introduction to Probability Theory and Stochastic Processes)

Time allowed: 1 hour **Minor 2 Examination** **Max. Marks: 25**

Name: Entry Number: Signature:

Multiple Selection Questions: **Section 1** **($1 \times 5 = 5$ marks)**

Each of the following questions 1 to 5 has four options out of which one or more options can be correct. Write A, B, C or D which corresponds to the correct option for the first correct answer followed by space and the next correct answer and so on. **1 mark** is awarded if all correct answers are written, **0 mark** for no answer or partial correct answers or any incorrect answer.

1. Let $X_1 \sim Exp(3)$ and $X_2 \sim Exp(5)$ be two independent random variables. The distribution of $X_{(1)} = \min\{X_1, X_2\}$ is
(A) Exp (8) (B) Gamma (8,3/5) (C) Erlang (8, 3/5) (D) Exp (3/5) Answer: A

2. Let X and Y be continuous type random variables with joint pdf given by
$$f(x,y) = \frac{1}{2\pi}e^{-\frac{(x^2+y^2)}{2}}, \quad -\infty < x < \infty, -\infty < y < \infty.$$
Then, the correlation coefficient between X and Y is (A) 0.5 (B) 0 (C) -1 (D) 1 Answer: B

3. Let (X, Y) be a two-dimensional random variables with joint pdf $f(x, y)$. Then, the pdf of $U = X + Y$ is (A) $g(u) = \int_{-\infty}^{\infty} f(u-v, u-v)dv$ (B) $g(u) = \int_{-\infty}^{\infty} f(u, u-v)dv$
(C) $g(u) = \int_{-\infty}^{\infty} f(v, u-v)dv$ (D) $g(u) = \int_{-\infty}^{\infty} f(u-v, v)dv$
Answer: C, D

4. Let N be a positive integer random variable and $X_1, X_2, \ldots$ be a sequence of iid random variables. N is independent of X_i's. Let $S_N = \sum_{i=1}^{N} X_i$. Then, $Var(S_N)$ is
(A) $E(N)Var(X) + Var(N)E(X)$ (B) $E(N)Var(X) + Var(N)[E(X)]^2$
(C) $[E(N)]^2Var(X)+Var(N)[E(X)]^2$ (D) $E(N^2)Var(X)+Var(N)E(X^2)$ Answer: B

5. Let $(X_1, X_2, \ldots, X_n)$ be a n-dimensional random variables. Let $\sum$ be a $n \times n$ matrix where the entry in row i, column j, $\sum_{ij}$ is given by $Cov(X_i, X_j)$. If y is any $1 \times n$ vector, then
(A) $y\sum y^T \leq 0$ (B) $y\sum y^T < 0$ (C) $y\sum y^T \geq 0$ (D) $y\sum y^T > 0$ Answer: C

———————Rough Work ———————

Numeric Type Questions: **Section 2** **($5 \times 2 = 10$ marks)**

Write the answer upto 4 decimal places (D) or in fraction (F) or the expression (E) for the following questions 6 to 10. **2 marks** are awarded if answer is correct, and **0 mark** for no answer or an incorrect answer.

6. Let (X, Y) be a two-dimensional continuous type random variables with joint pdf $f(x, y) = \begin{cases} kx(x-y), & -x < y < x, 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$ Find k. Answer(F/D): $\frac{1}{8}$

7. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables such that $X_i \sim N(0, 1)$. Define $S_n = \sum_{i=1}^{n} X_i$, $n = 1, 2, \ldots$. Then, as $n \to \infty$, $\frac{S_n}{n}$ converges in probability to Answer(F/D): 0

8. Let (X, Y) be a two-dimensional continuous type random variables. Assume that, $E(X), E(Y)$ and $E(XY)$ are exist. Suppose that, $E(X \mid Y = y)$ does not depend on y. Find $E(XY)$. Answer(E): $E(X)\,E(Y)$

9. It is known that a IIT bus will arrive at random at Nilgiri hostel bus stop sometime between 8:30 A.M. and 8:45 A.M. Rahul decides that he will go at random to this location between these two times and will wait at most 5 minutes for the bus. If he misses it, he will take the cycle rickshaw. What is the probability that he will take the cycle rickshaw? Answer(F/D): 13/18

10. Let (X, Y) be a two-dimensional continuous type random variables with joint pdf

$$f(x, y) = xe^{-x(y+1)} \text{ for } x, y > 0 \text{ and } f(x, y) = 0 \text{ otherwise.}$$

Define $Z = XY$. Then, for $z \geq 0$, $P(Z \leq z)$ is given by: Answer(E): $1 - e^{-z}$

Rough Work

Subjective Type Questions: **Section 3** ($2 \times 5 = 10$ **marks**)

Write the answer in the same page provided for the questions 11 and 12. **Full marks** are awarded if all the steps are correct, and **partial marks** for an incorrect answer.

11. Let (X, Y) be a two-dimensional discrete type random variables with joint pmf $p(x, y) = cxy$ for $x = 1, 2, 3$ and $y = 1, 2, 3$ and equals zero otherwise.

    (a) Find c. (b) Find $P(1 \leq X \leq 2, Y \leq 2)$. (c) $P(Y = 3)$. ($1 + 2 + 2$ marks)

Answer:

a.) $\sum_x \sum_y p(x,y) = \sum_x \sum_y cxy = 1$

$\Rightarrow 1c + 2c + 3c + 2c + 4c + 6c + 3c + 6c + 9c = 1$

$\Rightarrow 36c = 1 \Rightarrow \boxed{c = 1/36}$ ✓

b) $P(1 \leq X \leq 2, Y \leq 2)$

$= P(X=1, Y=1) + P(X=1, Y=2) + P(X=2, Y=1) + P(X=2, Y=2) = \frac{1}{36}[1+2+2+4] = \frac{9}{36} = \boxed{\frac{1}{4}}$ ✓

c) $P(Y=3) = \sum_x P(X=x, Y=3)$

$= P(X=1, Y=3) + P(X=2, Y=3) + P(X=3, Y=3)$

$= \frac{1}{36}[3+6+9] = \frac{18}{36} = \boxed{\frac{1}{2}}$ ✓

12. Let $X_1, X_2, \ldots$ be iid random variables, each having pmf $P(X_i = 1) = \frac{7}{9} = 1 - P(X_i = 0)$. Let $Y_i = X_i + X_i^2$, $i = 1, 2, \ldots$. Use central limit theorem to evaluate $P\left(\sum_{i=1}^{30} Y_i > 60\right)$ approximately. Final answer can be in terms of $\Phi(z)$ where $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$.

(5 marks)

Answer:

$$E(x_i) = 1 \cdot \frac{7}{9} + 0 \cdot \frac{2}{9} = 7/9$$

$$\operatorname{var}(x_i) = E(x^2) - (E(x))^2$$

$$E(x_i^2) = 1 \cdot 7/9 = 7/9$$

$$\operatorname{var}(x_i) = 7/9 - \frac{7 \times 7}{9 \times 9} = \frac{7}{9} \cdot \frac{2}{9} = \frac{14}{81}$$

$$Y_i = x_i + x_i^2$$

$$E(Y_i) = E(x_i + x_i^2) = E(x_i) + E(x_i^2) = \frac{14}{9}$$

$$E(Y_i^2) = E((x_i + x_i^2)^2) = E(x_i^2 + x_i^4 + 2x_i^3)$$

$$= 7/9 + 7/9 + 2 \cdot 7/9 = \frac{28}{9}$$

$$\operatorname{var}(Y_i) = \frac{28}{9} - \frac{14}{9} \cdot \frac{14}{9} = \frac{14}{9}\left[2 - \frac{14}{9}\right] = \frac{4}{9} \cdot \frac{14}{9}$$

Using CLT

$$P\left(\sum_{i=1}^{30} Y_i > 60\right) = P\left(\frac{\sum_{i=1}^{30} Y_i - \frac{30 \times 14}{9}}{\sqrt{30 \times \frac{4 \times 14}{9 \times 9}}} > \frac{60 - \frac{30 \times 14}{9}}{\sqrt{\frac{30 \times 4 \times 14}{9 \times 9}}}\right)$$

$$= P\left(Z > \frac{13.33}{4.554}\right) = P(Z > 2.93) = 1 - P(Z \le 2.93)$$

$$= \boxed{1 - \Phi(2.93)}$$