DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY DELHI
MAJOR TEST 2020-2021 FIRST SEMESTER
MTL 107(NUMERICAL METHODS AND COMPUTATION)

Time: 1 hour 30 minutes **Max. Marks: 40**

$**$ **Answer to each question should begin on a new page** $**$

1a. If $x, y,$ and z are machine numbers in a $32-$bit word length computer, what upper bound can be given for the relative roundoff error in computing $z(x+y)$? (2)

1b. Consider Newton's method for finding the positive square root of $a > 0$. Then derive the following, assuming $x_0 > 0, x_0 \neq \sqrt{a}$.

$$x_{n+1}^2 - a = \left[\frac{x_n^2 - a}{2x_n}\right]^2,$$

$n \geq 0$, and thus $x_n > \sqrt{a}$ for all $n > 0$. Also prove that the iterates $\{x_n\}$ are a strictly decreasing sequence for $n \geq 1$. (4)

2a. If $f[x, x_0, \ldots\ldots, x_k]$ is a polynomial (in x) of degree $m > 0$, then prove or disprove that $f[x, x_0, \ldots\ldots, x_{k+1}]$ is a polynomial of degree $m-1$. (2)

2b. The interpolating polynomial for the function $f(x)$ on the set of distinct points $x_0, x_1, \ldots\ldots, x_n$ is given as $P_n(x) = \sum_{k=0}^{n} l_k(x) f(x_k)$. Find an explicit expression for $\sum_{k=0}^{n} l_k(0) x_k^{n+1}$. (2)

2c. A function $f(x)$ is defined on [0,1] and $\mid f^{(m)}(x) \mid \leq m!$ for $m = 1, 2, \ldots\ldots$. Let $P_n(x)$ be the interpolating polynomial of $f(x)$ at the points $1, q, q^2, \ldots\ldots, q^n$ where $0 < q < 1$. Then prove or disprove that $\lim_{n\to\infty} P_n(0) = f(0)$. (3)

3. Make a least squares fit for

x	1	2	3	4	5
y(x)	0.5	2	4.5	8	12.5

using the function ax^b where a and b are constants. (4)

4. Consider the linear system $Ax = b$ with $\mid A \mid \neq 0$. Let δA be perturbation of A and assume $\|\delta A\| < \frac{1}{\|A^{-1}\|}$. Then examine whether $A + \delta A$ is nonsingular or not. And if we define δx implicitly by $(A + \delta A)(x + \delta x) = b$ then prove or disprove

$$\frac{\|\delta x\|}{\|x\|} \leq Cond.(A) \frac{\|\delta A\|}{\|A\|}(1 + O(\|\delta A\|)).$$

(4)

P.T.O.

5. Consider the linear system $Ax = b$ with

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{bmatrix},$$

Do the both Gauss-Jacobi and Gauss-Seidel iterative Methods converge/diverge?. Justify your answer. (6)

6. By use of repeated Richardson extrapolation, find $f'(1)$ from the following values:

x	0.6	0.8	0.9	1.0	1.1	1.2	1.4
f(x)	0.707178	0.859892	0.925863	0.984007	1.033743	1.074575	1.127986

Apply the approximate formula

$$f'(x_0) \simeq \frac{f(x_0+h) - f(x_0-h)}{2h}$$

with $h = 0.4, 0.2, 0.1$. (4)

7. Find a quadrature formula

$$\int_0^1 f(x) \frac{dx}{\sqrt{x(1-x)}} = \alpha_1 f(0) + \alpha_2 f(\frac{1}{2}) + \alpha_3 f(1)$$

which is exact for polynomials of highest possible degree. Use this formula to evaluate

$$\int_0^1 \frac{dx}{\sqrt{(x-x^3)}}.$$

(5)

8. Consider the initial value problem $y'(x) = xy,\ y(0) = 1$. Estimate the error at $x = 1$, when Euler's method is used, with step size $h = 0.01$

(Hint: $|\ e_n\ |=|\ y(x_n) - y_n\ | \leq [\frac{e^{L(x_n-a)}-1}{L}] M\frac{h}{2}$ when Euler method is applied to the problem $y'(x) = f(x,y); y(a) = A$, in $a \leq x \leq b$ and $h = (b-a)/N,\ x_k = a + kh$ and $|\frac{\partial f}{\partial y}| \leq L,\ \ |y''(x)| \leq M$.) (4)