Name: ______________________

Gradescope Student ID: ______________________-

Entry No: ______________________-

MTL 107: 1st Semester 2021-22

Numerical Methods and Computation

Template for the Major Exam

Total Marks: 50

Exam Time: 10:00 to 12:00 on 22nd November

Venue: LHC 121

Instructions For Online Submission

(1) **Please write you name, Entry Number and Gradescope Student id properly. This is important. Normally gradescope student id is your IIT Delhi email username.**
(2) Total duration is 2Hrs. This include time to scan and upload. **Absolutely no additional time will be given.** So, it expected that you start uploading well before the deadline.
(3) This is a tempelate based exam submission and this is the template for the Major exam. Please take the print out of these pages as you have to write the solution on them only. **If you gave exam on other sheets, your submission will not be accepted.**
You have to upload both sides for all the pages including the empty pages(even the pages you did not write anything). In Total 18 pages should be uploaded.
(4) The exam will be proctored. Hence you need to keep your camera on during whole exam.
(5) In each problem, write down the steps of the solution clearly. Marks will not be granted if proper justification is not provided.

Question 1:[6 marks] Show that the fourth order Runge-Kutta method when applied to the differential equation $y' = \lambda y$ can be written in the form

$$y_{i+1} = \left(1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{(h\lambda)^3}{6} + \frac{(h\lambda)^4}{24}\right) y_i.$$

Question 1(cont...):

Question 2:[6 marks] Given the equations

$$y_1{}'(x) = x + y_1 + y_2, \quad y_1(0) = 1,$$
$$y_2{}'(x) = 1 + y_1 + y_2, \quad y_2(0) = -1,$$

estimate the value of $y_1(0.1)$ and $y_2(0.1)$ using Heun's method taking $h = 0.1$.

Question 2(cont...):

Question 3:[6 marks] A fifth-degree polynomial $P(x)$ satisfies $\Delta^5 P(0) = 120$, $\Delta^4 P(0) = -24$ and $\Delta^3 P(0) = 0$ where $\Delta P(x) = P(x+1) - P(x)$. Compute $\Delta^3 P(3)$.

Question 3(cont...):

Question 4: [6 marks] Determine α and β in the formula

$$\int_a^b f(x)\, dx = h \sum_{i=0}^{n-1} [f(x_i) + \alpha h f'(x_i) + \beta h^2 f''(x_i)] + O(h^m),$$

with the integer m as large as possible. Here $x_i, 0 \le i \le n$ are equally spaced points with step size h and $x_0 = a,\ x_n = b$.

Question 4(cont...):

Question 5: [7 Marks] Consider the linear system:

$$\begin{aligned} 2x_1 - x_2 &= 4, \\ -x_1 + 2x_2 - x_3 &= 1, \\ -x_2 + 2x_3 &= 2. \end{aligned}$$

Find the optimal ω for the above linear system to use in successive over relaxation (SOR). Explain the result you use to get the optimal value of ω. Use it to calculate two iteration with SOR method, starting with zero vector as initial guess.

Question 5(cont...):

Question 5(cont...):

Question 6: **[6 marks]** Let $l_j(x),\ j = 0, 1, ..., n$ denote the Lagrange's basis polynomials of degree n for distinct nodes $x_0, x_1, ..., x_n$. Let p be a non-negative integer. For $n = 5$,

(i) Show that,

$$x_0^p\ l_0(x) + x_1^p\ l_1(x) + x_2^p\ l_2(x) + x_3^p\ l_3(x) + +x_4^p\ l_4(x) + +x_5^p\ l_5(x) = x^p, \quad p \leq 5.$$

(ii) Explain why

$$x_0^p\ l_0(x) + x_1^p\ l_1(x) + x_2^p\ l_2(x) + x_3^p\ l_3(x) + x_4^p\ l_4(x) + x_5^p\ l_5(x) \neq x^p, \quad p > 5, x \in \mathbb{R}.$$

Question 6(cont...):

Question 7: [6 marks] Let $f(x) = 1 + x^4$. Use the data $(x_i, f(x_i)), i = 0, 1, 2, 3, 4$ with $x_0 = 0, x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 1$ to compute the divided difference table. Thereby, use the computed divided difference table to find the fourth order divided difference of $f(x)$.

Question 7(cont...):

Question 8: **[7 Marks]** Let A be a symmetric positive definite matrix. Then show that following statements are equivalent:
Statement 1: $\vec{x}^*$ is the solution of $A\vec{x} = \vec{b}$.
Statement 2: $\vec{x}^*$ is the unique minimizer of

$$\phi(\vec{x}) = \frac{1}{3}\vec{x}^\top A\vec{x} - \frac{2}{3}\vec{x}^\top \vec{b}.$$

Here $\vec{x}^\top$ denotes transpose of the vector $\vec{x}$.

Question 8(cont...):