MTL-108: Major Exam on 07/05/2018

Max Time: 2 hrs.

Max Marks: 50

-20 marks will be awarded with no warning if found using any wrong practice in exam

1. Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$ population. Using moment generating function approach, prove that the sample mean $\bar{X}$ and the random vector $(X_1 - \bar{X}, \ldots, X_n - \bar{X})$ are independent. [6]

2. A random variable $X$ follows the pdf $f(x) = \begin{cases} \dfrac{1}{x^2} & x > 1 \\ 0 & \text{otherwise} \end{cases}$

   Find the pdf of the random variable $Y = 1 - |X - 2|$. [7]

3. Let $X$ and $Y$ be two independent random variables both following exponential distribution

$$f(\xi) = \lambda e^{-\lambda \xi}, \quad \xi \geq 0,$$

   where $\lambda > 0$. Let $Z = X + Y$.

   Find the joint pdf of $X$ and $Z$ by method of transformation. Use this joint pdf to find the conditional pdf of $X$ given the realization of $Z$ is $z = 10$, that is, the expression for $f_{X|Z}(X = x \mid z = 10)$. [7]

4. Let $(X_1, \ldots, X_n)$ be a random sample from the population having a cumulative distribution function (cdf)

$$F_X(x) = 1 - \left(\frac{\theta}{x}\right)^{\beta}, \quad x \geq \theta,$$

   where $\theta > 0$, $\beta > 0$. Find the maximum likelihood estimator of $(\theta, \beta)$. [7]

5. Let $X$ be a random variable following a scaled $t$-distribution with degree of freedom $\nu = 5$ with pdf given by

$$f_X(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\theta\sqrt{\pi\nu}\ \Gamma(\frac{\nu}{2})}\left(1 + \frac{x^2}{\nu\theta^2}\right)^{-(\nu+1)/2}, \quad x \in \mathbb{R},$$

   where $\theta > 0$.

   Let $X_1, \ldots, X_n$ be a random sample chosen from the above population with observations $x_1, \ldots, x_n$, respectively. Prove that $\frac{3}{5n}\sum_{i=1}^{n} x_i^2$ is an unbiased estimator of $\theta^2$ for any $n \in \mathbb{N}$ (set of natural numbers). [7]

   (Hint (if you want to use): relate $X$ with standard $t$-distributed variate; then relate it with $F$-distributed variate and finally use expectation of $F$-distribution)

6. Consider a population following normal $N(\mu, 256)$ distribution. The critical region for testing null hypothesis $H_0 : \mu = 5$ against the alternative hypothesis $H_1 : \mu = k$ is given as $\bar{X} > k - 2$, where $\bar{X}$ denotes sample mean.

   Determine $k$ and the size $n$ of the random sample such that the probability of Type I error is 0.0228 and the probability of Type II error is 0.1587.? [6]

7. Let $X_1, \ldots, X_n$ be a random sample of size from a distribution with pdf

$$f(x, \beta) = \frac{x^6\, e^{-(x/\beta)}}{\beta^7\, \Gamma(7)}, \quad x > 0$$

   where $\beta > 0$. Using likelihood ratio, determine the most powerful critical region in terms of sample mean statistics $\bar{X}$ for testing the null hypothesis $H_0 : \beta = 5$ against $H_1 : \beta \neq 5$. [6]

8. A random sample $X_1, \ldots, X_6$ is taken from a normally distributed population with pdf $N(3, \sigma^2)$. We wish to test whether the variance $\sigma^2$ can be considered to be at least 5, against the hypothesis that it is less than 5. It is given that $\sum_{i=1}^{6}(x_i - 3)^2 = 15.75$. Perform the relevant hypothesis test at 1% significance level and conclude. [4]