Department of Mathematics

MTL 108 (Introduction to Statistics)

Major Exam

Time: 2 hour 15 minutes **Date: 15/05/21**
Max. Marks: 50

Note: The exam is closed-book, and all the questions are compulsory.

1. Let T be a closed triangle in the plane with vertices $(0,0)$, $(0,\sqrt{2})$, and $(\sqrt{2},\sqrt{2})$. Let the joint CDF $F(x,y)$ of random vector (X,Y) denote the area of intersection of T with $\{(x_1,x_2) \mid x_1 \le x, x_2 \le y\}$. Find the $F(x,y)$ and joint PDF of (X,Y). What are the marginal PDFs of X and Y ?

(6 marks)

2. Tom and Harry solve a probability question independently. The time (in minutes) for Tom to solve the question has a probability density function given by
$$f(x) = \begin{cases} 5e^{-5x}, & \text{if } x \ge 0 \\ 0 \text{ otherwise,} \end{cases}$$
and the time (in minutes) for Harry to solve the question has a probability density function given by
$$g(y) = \begin{cases} 3e^{-3y}, & \text{if } y \ge 0 \\ 0 \text{ otherwise.} \end{cases}$$
Then,
   (a) What is the probability that Tom solves the question before Harry?
   (b) Given that Tom requires more than 1 minute, find the probability that he finishes the question before Harry.
   (c) What is the probability that one of them finishes the question a minute or more before the other one?

(6 marks)

3. Let $(X_n)_{n=1}^{\infty}$ be a sequence of independent standard normal random variables.
   - How should $P(X_1^2 + X_2^2 + \cdots X_n^2 \le x)$ be approximated in terms of $\phi(x)$, where $\phi(x)$ is the CDF of standard normal distribution.
   - Using above approximation find i) $P(X_1^2 + X_2^2 + \cdots X_{100}^2 \le 120)$, ii) $P(80 \le X_1^2 + X_2^2 + \cdots X_{100}^2 \le 120)$.
   - Find c such that $P(X_1^2 + X_2^2 + \cdots X_{100}^2 \le 100 + c) = 0.95$.
   - Find c such that $P(100 - c \le X_1^2 + X_2^2 + \cdots X_{100}^2 \le 100 + c) = 0.95$.

(7 marks)

4. Let $X_1, \ldots, X_n$ be a sample from a normal population with unknown mean μ and unknown variance σ^2. Determine the maximum likelihood estimators (MLEs) of μ and σ^2. Are the MLEs unbiased estimators? Justify your claim with proper arguments.

(7 marks)

5. Let $X_1, \ldots, X_n, X_{n+1}$ be a sample from a normal population having an unknown mean μ and variance 1. Let $\bar{X}_n = \sum_{i=1}^{n} \frac{X_i}{n}$ be the average of the first n of them.
(a) What is the distribution of $X_{n+1} - \bar{X}_n$? (b) If $\bar{X}_n = 4$, give an interval that, within 90 percent confidence, will contain the value of X_{n+1}.

(5 marks)

6. Let $X_1, X_2, \cdots, X_n$ be a random sample from a normal population with known mean μ and unknown variance σ^2. Derive a test at α-level of significance for the following hypothesis

$H_0 : \sigma^2 \leq \sigma_0^2$

against the alternate hypothesis

$H_1 : \sigma^2 > \sigma_0^2$

where σ_0^2 is a specified constant. What is the p-value of above hypothesis? Show that the test has the property that the null hypothesis H_0 will be accepted for all values $\alpha \leq p$.

(8 marks)

7. (a) Let $X_1, X_2, \cdots, X_n$ be a random sample from a normal population with mean μ_1 and variance σ_1^2, and $Y_1, Y_2, \cdots, Y_m$ be a random sample from a normal population with mean μ_2 and variance σ_2^2 . Both the samples are independent from each other. The variances σ_1^2 and σ_2^2 are unknown but they are equal, i.e., $\sigma_1^2 = \sigma_2^2 = \sigma^2$. Derive a test at α-level of significance for the following hypothesis

   $H_0 : \mu_1 \leq \mu_2$

   against the alternate hypothesis

   $H_1 : \mu_1 > \mu_2$.

   What is the p-value of above hypothesis? Show that the test has the property that the null hypothesis H_0 will be accepted for all values $\alpha \leq p$.

   (b) It is argued that the mean resistance of wire A is greater than the mean resistance of wire B. You make the tests on each wire independently with the following results

Wire A	Wire B
0.140	0.135
0.138	0.140
0.143	0.136
0.142	0.142
0.144	0.138
0.137	0.140

   It is known that the samples are coming from normal population and the variances in resistance of both wires are unknown but equal. What conclusions can you drawn by applying the hypothesis test derived in part (a)?

(8+3 marks)

TABLE A1 *Standard Normal Distribution Function:* $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

TABLE A2 *Values of* $x^2_{\alpha,n}$

n	$\alpha = .995$	$\alpha = .99$	$\alpha = .975$	$\alpha = .95$	$\alpha = .05$	$\alpha = .025$	$\alpha = .01$	$\alpha = .005$
1	.0000393	.000157	.000982	.00393	3.841	5.024	6.635	7.879
2	.0100	.0201	.0506	.103	5.991	7.378	9.210	10.597
3	.0717	.115	.216	.352	7.815	9.348	11.345	12.838
4	.207	.297	.484	.711	9.488	11.143	13.277	14.860
5	.412	.554	.831	1.145	11.070	12.832	13.086	16.750
6	.676	.872	1.237	1.635	12.592	14.449	16.812	18.548
7	.989	1.239	1.690	2.167	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.484	36.415	39.364	42.980	45.558
25	10.520	11.524	13.120	14.611	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	40.113	43.194	46.963	49.645
28	12.461	13.565	15.308	16.928	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	42.557	45.772	49.588	52.336
30	13.787	14.953	16.791	18.493	43.773	46.979	50.892	53.672

Other Chi-Square Probabilities:

$x^2_{.9,9} = 4.2$ $P\{x^2_{16} < 14.3\} = .425$ $P\{x^2_{11} < 17.1875\} = .8976.$

TABLE A3 *Values of* $t_{\alpha,n}$

n	$\alpha = .10$	$\alpha = .05$	$\alpha = .025$	$\alpha = .01$	$\alpha = .005$
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.474	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
∞	1.282	1.645	1.960	2.326	2.576

Other t Probabilities:
$P\{T_8 < 2.541\} = .9825$ $P\{T_8 < 2.7\} = .9864$ $P\{T_{11} < .7635\} = .77$ $P\{T_{11} < .934\} = .81$ $P\{T_{11} < 1.66\} = .94$ $P\{T_{12} < 2.8\} = .984$.