Department of Mathematics

MTL 108 (Introduction to Statistics)

Minor Exam

Time: 1 hour 45 minutes **Date: 18/03/21**
Max. Marks: 25

Note: The exam is closed-book, and all the questions are compulsory.

1. Suppose you roll two fair, six-sided dice, one of which is red and the other of which is green. Define the following random variables:

   $X =$ The number shown on the red die,

   $$Y = \begin{cases} 0 & \text{if the two dice show the same number} \\ 1 & \text{if the number on the green die is bigger than the number on the red die} \\ 2 & \text{if the number on the red die is bigger than the number on the green die} \end{cases}$$

   a) Write down a table showing the joint probability mass function for X and Y.
   b) Compute $E(X)$ and $E(Y)$.
   c) Are X and Y independent?

   (3 marks)

2. Consider a probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = \{(u, v) \in \mathbb{R}^2 | 0 \leq u \leq 1, 0 \leq v \leq 1\}$, and $\mathcal{F}$ is a Borel σ-field on Ω, and $P(A) = \frac{\text{area of } A}{\text{area of } \Omega}$ for every $A \in \mathcal{F}$. Define a random variable $X : \Omega \to \mathbb{R}$ such that $X(u, v) = u + v$ for all $(u, v) \in \Omega$. Find the probability density function of X.

   (4 marks)

3. (*a*) Suppose an experiment having r possible outcomes $1, 2, 3, \ldots, r$ that occur with probabilities $p_1, p_2, \ldots, p_r$ is repeated n times independently. Let X be the number of times the first outcome occurs, and let Y be the number of times the second outcome occurs. Then, compute the correlation coefficient $\rho(X, Y)$.

   (*b*) Consider an experiment having three possible outcomes that occur with probabilities p_1, p_2, and p_3, respectively. Suppose two independent repetitions of the experiment are made and let X_i denote the number of times the ith outcome occurs.
   i) What is the PMF of $X_1 + X_2$?
   ii) Find $P(X_2 = y | X_1 + X_2 = 2)$, $y = 0, 1, 2$. (For any two events A and B, $P(A|B) = \frac{P(A \cap B)}{P(B)}$)

   (5 marks)

4. Let $X_1, X_2, \cdots, X_n$ be independent random variables having the common density function

   $$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

   Find (a) $E[\max(X_1, X_2, \cdots, X_n)]$ and (b) $E[\min(X_1, X_2, \cdots, X_n)]$.

   (4 marks)

5. Let $X_1,\ X_2$ be independent random variables with common probability density function given by

$$f(x) = \begin{cases} 1, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise} \end{cases}.$$

Find the PDF of $X_1 - X_2$.

(5 marks)

6. Let X be a nonnegative integer-valued random variable. Consider a function $g(t) = E(t^X)$ which is finite for all t and let x_0 be a positive number. Verify the following inequalities:
a) $P(X \leq x_0) \leq \frac{g(t)}{t^{x_0}}, \quad 0 \leq t \leq 1;$
b) $P(X \geq x_0) \leq \frac{g(t)}{t^{x_0}}, \quad t \geq 1.$

(4 marks)